

SYMPLECTIC RESOLUTIONS OF QUIVER VARIETIES AND CHARACTER VARIETIES

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ABSTRACT. In this article, we consider Nakajima quiver varieties from the point of view of symplectic algebraic geometry. Namely, we consider the question of when a quiver variety admits a projective symplectic resolution. A complete answer to this question is given. We also show that the smooth locus of a quiver variety coincides with the locus of θ -canonically stable points, generalizing a result of Le Bruyn. An interesting consequence of our results is that not all symplectic resolutions of quiver varieties appear to come from variation of GIT.

In the final part of the article, we consider the G -character variety of a compact Riemann surface of genus $g > 0$, when G is $\mathrm{SL}(n, \mathbb{C})$ or $\mathrm{GL}(n, \mathbb{C})$. We show that these varieties admit symplectic singularities. When the genus g is greater than one, we show that the singularities are terminal and locally factorial. As a consequence, these character varieties do not admit symplectic resolutions.

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1. INTRODUCTION

Nakajima's quiver varieties [34], [35], have become ubiquitous throughout representation theory. For instance, they play a key role in the categorification of representations of Kac-Moody Lie algebras, and the corresponding theory of canonical bases. They provide also étale-local models of singularities appearing in many important moduli spaces, together with, in most cases, a canonical symplectic resolution given by varying the stability parameter.

Surprisingly, there seems to be no explicit criterion in the literature for when a quiver variety admits a projective symplectic resolution (often, in applications, suitable sufficient conditions for

2010 *Mathematics Subject Classification.* 16S80, 17B63.

Key words and phrases. symplectic resolution, quiver variety, character variety, Poisson variety.

their existence are provided, but they do not appear always to be necessary). The purpose of this article is to give such an explicit criterion. Following arguments of Kaledin, Lehn and Sorger (who consider the surprisingly similar case of moduli spaces of semi-stable sheaves on a $K3$ or abelian surface), our classification result ultimately relies upon a deep result of Drezet on the local factoriality of certain GIT quotients.

Our classification begins by generalizing Crawley-Boevey's decomposition theorem [7] of affine quiver varieties into products of such varieties (let us call them irreducible for now), to the non-affine case (i.e., to quiver varieties with nonzero stability condition) (Theorem 1.3). Along the way, we also generalize Le Bruyn's [29, Theorem 3.2], which computes the smooth locus of these varieties, again from the affine to nonaffine setting (Theorem 1.13).

Then, our main result, Theorem 1.4, states that those quiver varieties admitting resolutions are exactly those whose irreducible factors, as above, are one of the following three types of varieties:

- (a) Varieties whose dimension vector are indivisible non-isotropic imaginary roots for the Kac-Moody Lie algebra associated to the quiver (so of dimension ≥ 4);
- (b) Symmetric powers of deformations or partial resolutions of du Val singularities (\mathbb{C}^2/Γ for $\Gamma < \mathrm{SL}_2(\mathbb{C})$);
- (c) Varieties whose dimension vector are twice an indivisible non-isotropic imaginary root whose Cartan pairing with itself is -2 .

The last type is the most interesting one, and is closely related to O'Grady's examples [30]: in this case, one cannot fully resolve or smoothly deform via a quiver variety, but after maximally smoothing in this way, the remaining singularities are étale-equivalent to the product of $V = \mathbb{C}^4$ with the locus of square-zero matrices in $\mathfrak{sp}(V)$ (which O'Grady considers, see also [30]), and the later locus is resolved by the cotangent bundle of the Lagrangian Grassmannian of V .

In the case of type (a), one can resolve or deform by varying the quiver parameters, whereas in the case of type (b), one cannot resolve in this way, but the variety is well-known to be isomorphic to another quiver variety (whose quiver is obtained by adding an additional vertex, usually called a framing, and arrows from it to the other vertices), which does admit a resolution via varying the parameters. Moreover, in this case, if the stability parameter is chosen to lie in the appropriate chamber, then the resulting resolution is a punctual Hilbert scheme of the minimal resolution of the original (deformed) du Val singularity.

1.1. Symplectic resolutions. In order to state precisely our main results, we recall some standard notation. Let $Q = (Q_0, Q_1)$ be a quiver with finitely many vertices and arrows. We fix a dimension vector $\alpha \in \mathbb{N}^{Q_0}$, deformation parameter $\lambda \in \mathbb{C}^{Q_0}$, and stability parameter $\theta \in \mathbb{Q}^{Q_0}$, such that $\lambda \cdot \alpha = \theta \cdot \alpha = 0$. Unless otherwise stated, we make the following assumption throughout the paper:

$$\text{If } \theta \neq 0 \text{ then } \lambda \in \mathbb{R}^{Q_0}. \tag{1}$$

Associated to this data is the (generally singular) variety, which Nakajima defined and called a “quiver variety,” see Section 2 for details,

$$\mathfrak{M}_\lambda(\alpha, \theta) := \mu^{-1}(\lambda)^\theta // G(\alpha).$$

Remark 1.1. The construction in [34, 35] is apparently more general, depending on an additional dimension vector, called the framing. However, as observed by Crawley-Boevey [6], every framed variety can be identified with an unframed one. In more detail, for the variety as in [34, 35] with framing $\beta \in \mathbb{N}^{Q_0}$, it is observed in [6, §1] that the resulting variety can alternatively be constructed by replacing Q by the new quiver $(Q_0 \cup \{\infty\}, \widetilde{Q}_1)$, where \widetilde{Q}_1 consists of Q_1 together with, for every $i \in Q_0$, β_i new arrows from ∞ to i ; then Nakajima’s β -framed variety is the same as $\mathfrak{M}_{(\lambda,0)}((\alpha, 1), (\theta, \sum_{i \in Q_0} -\theta_i))$. Thus, for the purposes of the questions addressed in this article, it is sufficient to consider the unframed varieties.

Let $R_{\lambda,\theta}^+$ denote those positive roots of Q that pair to zero with both λ and θ . If $\alpha \notin \mathbb{N}R_{\lambda,\theta}^+$ then $\mathfrak{M}_\lambda(\alpha, \theta) = \emptyset$, therefore we assume $\alpha \in \mathbb{N}R_{\lambda,\theta}^+$. Recall that a normal variety X is said to be a symplectic singularity if there exists a (algebraic) symplectic 2-form ω on the smooth locus of X such that $\pi^*\omega$ extends to a regular 2-form on the whole of Y , for any resolution of singularities $\pi : Y \rightarrow X$. We say that π is a symplectic resolution if $\pi^*\omega$ extends to a non-degenerate 2-form on Y .

Proposition 1.2. *The variety $\mathfrak{M}_\lambda(\alpha, \theta)$ is an irreducible symplectic singularity.*

From both the representation theoretic and the geometric point of view, it is important to know when the variety $\mathfrak{M}_\lambda(\alpha, \theta)$ admits a symplectic resolution. In this article, we address this question, giving a complete answer. The first step is to reduce to the case where α is a root for which there exists a θ -stable representation of dimension α of the deformed preprojective algebra $\Pi^\lambda(Q)$. This is done via the *canonical decomposition* of α , as described by Crawley-Boevey; it is analogous to Kac’s canonical decomposition. Associated to λ, θ is a set $\Sigma_{\lambda,\theta} \subset R^+$ (which we will define in §2 below). Then α admits a canonical decomposition

$$\alpha = n_1\sigma^{(1)} + \cdots + n_k\sigma^{(k)} \tag{2}$$

with $\sigma^{(i)} \in \Sigma_{\lambda,\theta}$ pairwise distinct, such that any other decomposition of α into a sum of roots belonging to $\Sigma_{\lambda,\theta}$ is a refinement of the decomposition (2). Crawley-Boevey’s Decomposition Theorem [7], which we will show holds in somewhat greater generality, then says that:

Theorem 1.3. *The symplectic variety $\mathfrak{M}_\lambda(\alpha, \theta)$ is isomorphic to $S^{n_1}\mathfrak{M}_\lambda(\sigma^{(1)}, \theta) \times \cdots \times S^{n_k}\mathfrak{M}_\lambda(\sigma^{(k)}, \theta)$ and $\mathfrak{M}_\lambda(\alpha, \theta)$ admits a projective symplectic resolution if and only if each $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ admits a projective symplectic resolution.*

Therefore it suffices to assume that $\alpha \in \Sigma_{\lambda,\theta}$. We write $\gcd(\alpha)$ for the greatest common divisor of the integers $\{\alpha_i\}_{i \in Q_0}$. The dimension vector α is said to be *divisible* if $\gcd(\alpha) > 1$. Otherwise, it

is indivisible. Our main theorem states the following. Let $p(\alpha) := 1 - \frac{1}{2}(\alpha, \alpha)$ where $(-, -)$ is the Cartan matrix associated to the undirected graph underlying the quiver, i.e., $(e_i, e_j) = 2 - |\{a \in Q_1 : a : i \rightarrow j \text{ or } a : j \rightarrow i\}|$ for elementary vectors e_i, e_j .

Theorem 1.4. *Let $\alpha \in \mathbb{N}R_{\lambda, \theta}^+$. The quiver variety $\mathfrak{M}_\lambda(\alpha, \theta)$ admits a projective symplectic resolution if and only if each non-isotropic imaginary root σ appearing in the canonical decomposition of α is either indivisible or $(\gcd(\sigma), p(\gcd(\sigma)^{-1}\sigma)) = (2, 2)$.*

If $\alpha \in \Sigma_{\lambda, \theta}$ is an indivisible non-isotropic imaginary root, then a projective symplectic resolution of $\mathfrak{M}_\lambda(\alpha, \theta)$ is given by moving θ to a generic stability parameter. However, this fails if α is a non-isotropic imaginary root such that $(\gcd(\alpha), p(\gcd(\alpha)^{-1}\alpha)) = (2, 2)$. It seems unlikely that $\mathfrak{M}_\lambda(\alpha, \theta)$ can be resolved by another quiver variety in this case. Instead, we show that the 10-dimensional symplectic singularity $\mathfrak{M}_\lambda(\alpha, \theta)$ can be resolved by blowing up a certain sheaf of ideals. Let θ' be a generic stability parameter such that $\theta' \geq \theta$; see section 2.4 for the definition of \geq .

Theorem 1.5. *There exists a sheaf of ideals \mathcal{I} on $\mathfrak{M}_\lambda(\alpha, \theta')$ such that if $\widetilde{\mathfrak{M}}_\lambda(\alpha, \theta')$ is the blowup of $\mathfrak{M}_\lambda(\alpha, \theta')$ along \mathcal{I} , then the canonical morphism $\pi : \widetilde{\mathfrak{M}}_\lambda(\alpha, \theta') \rightarrow \mathfrak{M}_\lambda(\alpha, \theta)$ is a projective symplectic resolution of singularities.*

Set-theoretically, the set of zeros of \mathcal{I} is precisely the singular locus of $\mathfrak{M}_\lambda(\alpha, \theta')$.

Remark 1.6. Since our reduction arguments are similar to those of [26], it is not so surprising (in hindsight at least) that Theorem 1.4 is completely analogous to [26, Theorems A & B]. In both cases, it is self-extensions of a certain kind that cannot be resolved symplectically.

1.2. Divisible non-isotropic imaginary roots. The real difficulty in the proof of Theorem 1.4 is in showing that if $\alpha \in \Sigma_{\lambda, \theta}$ is a divisible non-isotropic imaginary root such that

$$(\gcd(\alpha), p(\gcd(\alpha)^{-1}\alpha)) \neq (2, 2),$$

then $\mathfrak{M}_\lambda(\alpha, \theta)$ does not admit a projective symplectic resolution. Based upon a deep result of Drezet [13], who considered instead the moduli space of semi-stable sheaves on a rational surface, we show in Corollary 6.8 that

Theorem 1.7. *Assume that θ is generic. The quiver variety $\mathfrak{M}_\lambda(\alpha, \theta)$ is locally factorial.*

Since it is clear that $\mathfrak{M}_\lambda(\alpha, \theta)$ has terminal singularities, the above theorem implies that it cannot admit a projective symplectic resolution. In fact, we prove a more precise statement than Theorem 1.7, see Corollary 6.8, which does not require that θ be generic. From Corollary 6.8, we deduce that $\mathfrak{M}_\lambda(\alpha, \theta)$ does not admit a projective symplectic resolution. In fact, by the argument given in the proof of Theorem 6.13, we see that the corollary implies that this statement holds for open subsets of $\mathfrak{M}_\lambda(\alpha, \theta)$:

Corollary 1.8. *If $U \subseteq \mathfrak{M}_\lambda(\alpha, \theta)$ is any singular Zariski open subset, then U does not admit a symplectic resolution.*

In particular, for many choices of U , we obtain a variety which formally locally admits a symplectic resolution everywhere, but does not admit one globally. For example, if $\alpha = 2\beta$ for some $\beta \in \Sigma_{\lambda, \theta}$ with $p(\beta) \geq 3$ (cf. the definition of p above Theorem 1.4), then we can let U be the complement of the locus of representations X of the doubled quiver in $\mu_\alpha^{-1}(\lambda)$ which decompose as $X = Y^{\oplus 2}$ for Y a simple representation of dimension vector β .

Remark 1.9. One does not need the full strength of the above theorem to show that $\mathfrak{M}_\lambda(\alpha, \theta)$ does not admit a symplectic resolution: it suffices to show that a formal neighborhood of some point does not admit a symplectic resolution. This can actually be deduced from a result of Kaledin, Lehn, and Sorger: see Remark 3.5 below for details. However, this does not actually simplify the proof since those authors also appeal to Drezet’s result in the same way (which is indeed where we learned about it). Moreover, this would not be enough to imply Corollary 1.8.

There is one quiver variety in particular that captures the “unresolvable” singularities of $\mathfrak{M}_\lambda(\alpha, \theta)$. This variety, which we denote $\mathfrak{X}(n, d)$ with $n, d \in \mathbb{N}$, has been extensively studied in the works of Lehn, Kaledin and Sorger. Concretely,

$$\mathfrak{X}(n, d) := \left\{ (X_1, Y_1, \dots, X_d, Y_d) \in \text{End}_{\mathbb{C}}(\mathbb{C}^n) \mid \sum_{i=1}^d [X_i, Y_i] = 0 \right\} // \text{GL}(n, \mathbb{C}),$$

Viewed as a special case of Corollary 6.8, it is shown in [26] that

Theorem 1.10. *Let $n, d \geq 2$, with $(n, d) \neq (2, 2)$. The symplectic variety $\mathfrak{X}(n, d)$ is locally factorial and terminal. In particular, it has no projective symplectic resolution.*

When $d = 1$, the Hilbert scheme of n points in the plane provides a symplectic resolution of $\mathfrak{X}(n, d) \simeq S^n \mathbb{C}^2$; see [17, Theorem 1.2.1, Lemma 2.8.3]. When $n = 1$, one has $\mathfrak{X}(n, d) \simeq \mathbb{A}^{2d}$.

Remark 1.11. It is interesting to note that [6, Theorem 1.1] implies that the moment map

$$(X_1, Y_1, \dots, X_d, Y_d) \mapsto \sum_{i=1}^d [X_i, Y_i]$$

is flat when $d > 1$, in contrast to the case $d = 1$, which is easily seen not to be flat.

Remark 1.12. Generalizing the Geiseker moduli spaces that arise from framings of the Jordan quiver, it seems likely that the framed versions of $\mathfrak{X}(n, d)$ (which are smooth for generic stability parameters) should have interesting combinatorial and representation theoretic properties.

1.3. Smooth versus canonically-stable points. In order to decide when the variety $\mathfrak{M}_\lambda(\alpha, \theta)$ is smooth, we describe the smooth locus in terms of θ -stable representations. Write the canonical decomposition $n_1\sigma^{(1)} + \cdots + n_k\sigma^{(k)}$ of $\alpha \in \mathbb{N}R_{\lambda, \theta}^+$ as $\tau^{(1)} + \cdots + \tau^{(\ell)}$, where a given root $\tau \in \Sigma_{\lambda, \theta}$ may appear multiple times. Recall that a point $x \in \mathfrak{M}_\alpha(\lambda, \theta)$ is said to be polystable if it is a direct sum of θ -stable representations. We say that x is *θ -canonically stable* if $x = x_1 \oplus \cdots \oplus x_\ell$ where each x_i is θ -stable, $\dim x_i = \tau^{(i)}$ and $x_i \not\cong x_j$ for $i \neq j$. The set of θ -canonically stable points in $\mathfrak{M}_\lambda(\alpha, \theta)$ is a dense open subset. When $\theta = 0$, the result below is due to Le Bruyn [29, Theorem 3.2] (whose arguments we generalize).

Theorem 1.13. *A point $x \in \mathfrak{M}_\lambda(\alpha, \theta)$ belongs to the smooth locus if and only if it is θ -canonically stable.*

Remark 1.14. Theorem 1.13 confirms the expectation stated after Lemma 4.4 of [21].

An element $\sigma \in \Sigma_{\lambda, \theta}$ is said to be *minimal* if there are no $\beta^{(1)}, \dots, \beta^{(r)} \in \Sigma_{\lambda, \theta}$, with $r \geq 2$, such that $\sigma = \beta^{(1)} + \cdots + \beta^{(r)}$.

Corollary 1.15. *The variety $\mathfrak{M}_\lambda(\alpha, \theta)$ is smooth if, and only if, in the canonical decomposition $\alpha = n_1\sigma^{(1)} + \cdots + n_k\sigma^{(k)}$ of α , each $\sigma^{(i)}$ is minimal, and occurs with multiplicity one whenever $\sigma^{(i)}$ is imaginary.*

1.4. Namikawa's Weyl group. When both λ and θ are zero, $\mathfrak{M}_0(\alpha, 0)$ is an affine conic symplectic singularity. Associated to $\mathfrak{M}_0(\alpha, 0)$ is *Namikawa's Weyl group* W , a finite reflection group. In order to compute W , one needs to describe the codimension two symplectic leaves of $\mathfrak{M}_0(\alpha, 0)$. More generally, we consider the codimension two leaves in a general quiver variety $\mathfrak{M}_\lambda(\alpha, \theta)$. We show that these are parameterized by *isotropic decompositions* of α .

Definition 1.16. The decomposition $\alpha = \beta^{(1)} + \cdots + \beta^{(s)} + m_1\gamma^{(1)} + \cdots + m_t\gamma^{(t)}$ is said to be an isotropic decomposition if

- (1) $\beta^{(i)}, \gamma^{(j)} \in \Sigma_{\lambda, \theta}$.
- (2) The $\beta^{(i)}$ are *pairwise distinct* imaginary roots.
- (3) The $\gamma^{(i)}$ are *pairwise distinct* real roots.
- (4) If $\overline{Q''}$ is the quiver with $s+t$ vertices without loops and $-(\alpha^{(i)}, \alpha^{(j)})$ arrows between vertices $i \neq j$, where $\alpha^{(i)}, \alpha^{(j)} \in \{\beta^{(1)}, \dots, \beta^{(s)}, \gamma^{(1)}, \dots, \gamma^{(t)}\}$, then Q'' is an affine Dynkin quiver.
- (5) The dimension vector $(1, \dots, 1, m_1, \dots, m_t)$ of Q'' (where there are s one's) equals δ , the minimal imaginary root.

Theorem 1.17. *Let $\alpha \in \Sigma_{\lambda, \theta}$ be imaginary. Then the codimension two strata of $\mathfrak{M}_\lambda(\alpha, \theta)$ are in bijection with the isotropic decompositions of α .*

1.5. Character varieties. The methods we use seem to be applicable to many other situations. Indeed, as we have noted previously, they were first developed by Kaledin-Lehn-Soerger in the context of semi-stable sheaves on a $K3$ or abelian surface. Any situation where the symplectic singularity is constructed as a Hamiltonian reduction with respect to a reductive group of type A is amenable to this sort of analysis. One such situation, which is of crucial importance in geometric group theory, is that of character varieties of a Riemannian surface.

Let Σ be a compact Riemannian surface of genus $g > 0$ and π its fundamental group. The SL -character variety of Σ is the affine quotient

$$\mathcal{Y}(n, g) := \text{Hom}(\pi, \text{SL}(n, \mathbb{C})) // \text{SL}(n, \mathbb{C}).$$

If $g > 1$ then $\dim \mathcal{Y}(n, g) = 2(g-1)(n^2-1)$, and when $g = 1$, it has dimension $2(n-1)$. We do not consider the case where Σ has punctures, since the corresponding character variety is smooth in this case.

Theorem 1.18. *The variety $\mathcal{Y}(n, g)$ is an irreducible symplectic singularity.*

The same arguments, using Drezet's Theorem, that we have used to prove Theorem 1.7 are also applicable to the symplectic singularities $\mathcal{Y}(n, g)$. We show that:

Theorem 1.19. *Assume that $g > 1$ and $(n, g) \neq (2, 2)$. The variety $\mathcal{Y}(n, g)$ has locally factorial, terminal singularities.*

Arguing as in the proof of Theorem 6.13, Theorem 1.19 implies:

Corollary 1.20. *Assume $g > 1$ and $(n, g) \neq (2, 2)$. Then the symplectic singularity $\mathcal{Y}(n, g)$ does not admit a symplectic resolution. The same holds for any singular open subset.*

Remark 1.21. Parallel to Remark 1.9, we can give an alternative proof of the first statement of Corollary 1.20 using formal localization, reducing to the quiver variety case. The formal neighborhood of the identity of $\mathcal{Y}(n, g)$ is well known to identify with the formal neighborhood of $(0, \dots, 0)$ in the quotient

$$\left\{ (X_1, Y_1, \dots, X_d, Y_d) \in \mathfrak{sl}(n, \mathbb{C}) \mid \sum_{i=1}^d [X_i, Y_i] = 0 \right\} // \text{SL}(n, \mathbb{C}).$$

This is (essentially) the formal neighborhood of zero of the quiver variety $\mathfrak{M}_{(n)}(0, 0)$ for the quiver Q with one vertex and g arrows. Since $\mathfrak{M}_{(n)}(0, 0)$ is conical, as we recall in Lemma 6.15 below, it admits a symplectic resolution if and only if its formal neighborhood of zero does. But the fact that this does not admit a resolution when $g > 1$ and $(n, g) \neq (2, 2)$ is [26, Theorem B] (whose proof we generalize to prove Theorem 1.19). However, we cannot directly conclude Theorem 1.19 using formal localization, and neither the stronger last statement of Corollary 1.20.

Remark 1.22. We expect that, as pointed out after Corollary 1.8, one can obtain singular open subsets $U \subseteq \mathcal{Y}(n, g)$ in the case $g > 1$ and $(n, g) \neq (2, 2)$ for which the formal neighborhood of every point does admit a resolution, even though the entire U does not admit one by the corollary. Probably, one example is analogous to the one given there: for $n = 2$ and $g \geq 3$, and U the complement of the locus of representations of the form $Y^{\oplus 2}$ for Y one-dimensional (and hence irreducible). This would be such an example if Question 8.8 has a positive answer in the case $V = Y^{\oplus 2}$ (with $n = 2$ and $g \geq 3$).

Once again, the case of a genus two Riemann surface and 2-dimensional representations of π i.e. $(n, g) = (2, 2)$, is special. In this case $\mathcal{Y}(2, 2)$ does not have terminal singularities. Moreover, by work of Lehn and Sorger [30], $\mathcal{Y}(2, 2)$ does admit a symplectic resolution. In fact an explicit resolution can be constructed.

Theorem 1.23. *The blowup $\sigma : \tilde{\mathcal{Y}}(2, 2) \rightarrow \mathcal{Y}(2, 2)$ of $\mathcal{Y}(2, 2)$ along the reduced ideal defining the singular locus of $\mathcal{Y}(2, 2)$ defines a symplectic resolution of singularities.*

Remark 1.24. When $g = 1$, the barycentric Hilbert scheme $\text{Hilb}_0^n(\mathbb{C}^\times \times \mathbb{C}^\times)$ provides a resolution of singularities for $\mathcal{Y}(n, g)$.

In the body of the article, we consider instead the GL-character variety

$$\mathcal{X}(n, g) = \text{Hom}(\pi, \text{GL}(n, \mathbb{C})) // \text{GL}(n, \mathbb{C}).$$

In section 8.6, we deduce Theorems 1.18, 1.19 and 1.23, and Corollary 1.20, from the corresponding results for $\mathcal{X}(n, g)$. Similar techniques are applicable to the Hitchin's moduli spaces of semi-stable Higgs bundles over smooth projective curves. Details will appear in future work.

1.6. Acknowledgments. The first author was partially supported by EPSRC grant EP/N005058/1. The second author was partially supported by NSF Grant DMS-1406553. The authors are grateful to the University of Glasgow for the hospitality provided during the workshop “Symplectic representation theory”, where part of this work was done, and the second author to the 2015 Park City Mathematics Institute for an excellent working environment. We would like to thank Victor Ginzburg for suggesting we consider character varieties. We would also like to thank David Jordan, Johan Martens and Ben Martin for their many explanations regarding character varieties.

1.7. Proof of the main results. The proof of the theorems and corollaries stated in the introduction can be found in the following subsections.

| | |
|-----------------|---------------|
| Proposition 1.2 | : Section 6.3 |
| Theorem 1.3 | : Section 6.4 |
| Theorem 1.4 | : Section 6.4 |
| Theorem 1.5 | : Section 5 |
| Theorem 1.7 | : Section 6.2 |
| Corollary 1.8 | : Section 6.4 |
| Theorem 1.10 | : – |
| Theorem 1.13 | : Section 4 |
| Corollary 1.15 | : Section 4 |
| Theorem 1.17 | : Section 7 |
| Theorem 1.18 | : Section 8.6 |
| Theorem 1.19 | : Section 8.6 |
| Corollary 1.20 | : – |
| Theorem 1.23 | : Section 8.6 |

Throughout, variety will mean a reduced, quasi-projective scheme of finite type over \mathbb{C} . By *symplectic manifold*, we mean a smooth variety equipped with a non-degenerate closed *algebraic* 2-form.

2. QUIVER VARIETIES

In this section we fix notation.

2.1. Notation. Let $\mathbb{N} := \mathbb{Z}_{\geq 0}$. We work over \mathbb{C} throughout. All quivers considered will have a finite number of vertices and arrows. We *allow* Q to have loops at vertices. Let $Q = (Q_0, Q_1)$ be a quiver, where Q_0 denotes the set of vertices and Q_1 denotes the set of arrows. For a dimension vector $\alpha \in \mathbb{N}^{Q_0}$, $\text{Rep}(Q, \alpha)$ will be the space of representations of Q of dimension α . The group $G(\alpha) := \prod_{i \in Q_0} GL_{\alpha_i}(\mathbb{C})$ acts on $\text{Rep}(Q, \alpha)$; write $\mathfrak{g}(\alpha) = \text{Lie } G(\alpha)$. The torus \mathbb{C}^\times in $G(\alpha)$ of diagonal matrices acts trivially on $\text{Rep}(Q, \alpha)$. Thus, the action factors through $PG(\alpha) := G(\alpha)/\mathbb{C}^\times$.

Let \overline{Q} be the doubled quiver so that there is a natural identification $T^*\text{Rep}(Q, \alpha) = \text{Rep}(\overline{Q}, \alpha)$. The group $G(\alpha)$ acts symplectically on $\text{Rep}(\overline{Q}, \alpha)$ and the corresponding moment map is $\mu : \text{Rep}(\overline{Q}, \alpha) \rightarrow \mathfrak{g}(\alpha)$, where we have identified $\mathfrak{g}(\alpha)$ with its dual using the trace form. An element $\lambda \in \mathbb{C}^{Q_0}$ is identified with the tuple of scalar matrices $(\lambda_i \text{Id}_{V_i})_{i \in Q_0} \in \mathfrak{g}(\alpha)$. The affine quotient $\mu^{-1}(\lambda)/G(\alpha)$ parameterizes semi-simple representations of the *deformed preprojective algebra* $\Pi^\lambda(Q)$; see [6] for details.

If M is a finite dimensional $\Pi^\lambda(Q)$ -module, then $\dim M$ will always denote the dimension *vector* of M , and not just its total dimension.

2.2. Root systems. The coordinate vector at vertex i is denoted e_i . The set \mathbb{N}^{Q_0} of dimension vectors is partially ordered by $\alpha \geq \beta$ if $\alpha_i \geq \beta_i$ for all i and we say that $\alpha > \beta$ if $\alpha \geq \beta$ with $\alpha \neq \beta$.

Following [8, Section 8], the vector α is called *sincere* if $\alpha_i > 0$ for all i . The Ringel form on \mathbb{Z}^{Q_0} is defined by

$$\langle \alpha, \beta \rangle = \sum_{i \in I} \alpha_i \beta_i - \sum_{a \in Q} \alpha_{t(a)} \beta_{h(a)}.$$

Let $(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$ denote the corresponding Euler form and set $p(\alpha) = 1 - \langle \alpha, \alpha \rangle$. The fundamental region $\mathcal{F}(Q)$ is the set of $0 \neq \alpha \in \mathbb{N}^{Q_0}$ with connected support and with $(\alpha, e_i) \leq 0$ for all i .

If i is a loopfree vertex, so $p(e_i) = 0$, there is a reflection $s_i : \mathbb{Z}^{Q_0} \rightarrow \mathbb{Z}^{Q_0}$ defined by $s_i(\alpha) = \alpha - (\alpha, e_i)e_i$. The real roots (respectively imaginary roots) are the elements of \mathbb{Z}^{Q_0} which can be obtained from the coordinate vector at a loopfree vertex (respectively \pm an element of the fundamental region) by applying some sequence of reflections at loopfree vertices. Let R^+ denote the set of positive roots. Recall that a root β is *isotropic* imaginary if $p(\beta) = 1$ and *non-isotropic* imaginary if $p(\beta) > 1$. We say that a dimension vector α is *indivisible* if the greatest common divisor of the α_i is one.

2.3. The canonical decomposition. In this section we recall the *canonical decomposition* defined by Crawley-Boevey (not to be confused with Kac's canonical decomposition). Fix $\lambda \in \mathbb{C}^{Q_0}$ and $\theta \in \mathbb{Q}^{Q_0}$. Then $R_{\lambda, \theta}^+ := \{\alpha \in R^+ \mid \lambda \cdot \alpha = \theta \cdot \alpha = 0\}$. Following [6], we define

$$\Sigma_{\lambda, \theta} = \left\{ \alpha \in R_{\lambda, \theta}^+ \mid p(\alpha) > \sum_{i=1}^r p(\beta^{(i)}) \text{ for any decomposition } \alpha = \beta^{(1)} + \dots + \beta^{(r)} \text{ with } r \geq 2, \beta^{(i)} \in R_{\lambda, \theta}^+ \right\}.$$

Choosing a parameter $\lambda' \in \mathbb{C}^{Q_0}$ such that $R_{\lambda, \theta}^+ = R_{\lambda'}^+$, [7, Theorem 1.1] implies that¹

Proposition 2.1. *Let $\alpha \in \mathbb{N}R_{\lambda, \theta}^+$. Then α admits a unique decomposition $\alpha = n_1\sigma^{(1)} + \dots + n_k\sigma^{(k)}$ as a sum of element $\sigma^{(i)} \in \Sigma_{\lambda, \theta}$ such that any other decomposition of α as a sum of elements from $\Sigma_{\lambda, \theta}$ is a refinement of this decomposition.*

The following elementary fact will be used frequently.

Lemma 2.2. *Let α be a non-isotropic imaginary root and $m \in \mathbb{N}$. Then $m\alpha \in \Sigma_{\lambda, \theta}$ if and only if $\alpha \in \Sigma_{\lambda, \theta}$.*

Proof. Assume that $m\alpha \in \Sigma_{\lambda, \theta}$. If $\alpha \notin \Sigma_{\lambda, \theta}$ then $\alpha = \beta^{(1)} + \dots + \beta^{(r)}$ with $r \geq 2$ and $p(\alpha) \leq \sum_i p(\beta^{(i)})$. We have

$$p(m\alpha) = 1 - \langle m\alpha, m\alpha \rangle = mp(\alpha) - (m-1) \leq \sum_i mp(\beta^{(i)}) - (m-1) \leq \sum_i mp(\beta^{(i)})$$

which implies that $m\alpha \notin \Sigma_{\lambda, \theta}$. Conversely, if $\alpha \in \Sigma_{\lambda, \theta}$ then [7, Proposition 1.2 (3)] says that $m\alpha \in \Sigma_{\lambda, \theta}$. \square

¹We don't have to choose such a λ' , since the arguments of [7] can be simply generalized to the context of the pair (θ, λ) .

2.4. Stability. Let $\theta \in \mathbb{Q}^{Q_0}$ be a rational stability condition. Recall that a $\Pi^\lambda(Q)$ -representation M , such that $\theta(M) = 0$, is said to be θ -stable, respectively θ -semi-stable, if $\theta(M') < 0$, respectively $\theta(M') \leq 0$, for all proper non-zero subrepresentations M' of M . A representation M is said to be θ -polystable if $M = M_1 \oplus \cdots \oplus M_k$ with $\theta(M_i) = 0$, such that each M_i is θ -stable. The set of θ -semi-stable points in $\mu^{-1}(\lambda)$ is denoted $\mu^{-1}(\lambda)^\theta$. We define a partial order on \mathbb{Q}^{Q_0} by setting $\theta' \geq \theta$ if M θ -semistable implies that M is θ' -semi stable, i.e.,

$$\theta' \geq \theta \iff \mu^{-1}(\lambda)^{\theta'} \subset \mu^{-1}(\lambda)^\theta.$$

The space $\text{Rep}(\overline{Q}, \alpha)$ has a natural Poisson structure. Since the action of $G(\alpha)$ on $\text{Rep}(\overline{Q}, \alpha)$ is Hamiltonian,

$$\mathfrak{M}_\lambda(\alpha, \theta) = \mu^{-1}(\lambda)^\theta // G(\alpha) := \text{Proj} \bigoplus_{k \geq 0} \mathbb{C} [\mu^{-1}(\lambda)^\theta]^{k\theta}$$

is a Poisson variety.

Lemma 2.3. *If $\theta' \geq \theta$, then there is a projective Poisson morphism $\mathfrak{M}_\lambda(\alpha, \theta') \rightarrow \mathfrak{M}_\lambda(\alpha, \theta)$.*

Proof. By definition, we have a $G(\alpha)$ -equivariant embedding $\mu^{-1}(\lambda)^{\theta'} \hookrightarrow \mu^{-1}(\lambda)^\theta$. This induces a morphism

$$\mathfrak{M}_\lambda(\alpha, \theta') = \mu^{-1}(\lambda)^{\theta'} // G(\alpha) \longrightarrow \mu^{-1}(\lambda)^\theta // G(\alpha) = \mathfrak{M}_\lambda(\alpha, \theta),$$

between geometric quotients. We need to show that this morphism is projective. This is local on $\mathfrak{M}_\lambda(\alpha, \theta)$. Therefore we may choose $n \gg 0$ and a $n\theta$ -semi-invariant f and consider the open subsets $U \cap \mu^{-1}(\lambda)^{\theta'}$ and $U \cap \mu^{-1}(\lambda)^\theta$, where $U = (f \neq 0) \subset \text{Rep}(\overline{Q}, \alpha)$. Then $(U \cap \mu^{-1}(\lambda)^\theta) // G(\alpha) = \text{Spec } \mathbb{C} [U \cap \mu^{-1}(\lambda)^\theta]^{G(\alpha)}$ is an open subset of $\mathfrak{M}_\lambda(\alpha, \theta)$ and

$$(U \cap \mu^{-1}(\lambda)^{\theta'}) // G(\alpha) = \text{Proj} \bigoplus_{k \geq 0} \mathbb{C} [U \cap \mu^{-1}(\lambda)^{\theta'}]^{k\theta'}$$

such that $(U \cap \mu^{-1}(\lambda)^{\theta'}) // G(\alpha) \rightarrow (U \cap \mu^{-1}(\lambda)^\theta) // G(\alpha)$ is the projective morphism

$$\text{Proj} \bigoplus_{k \geq 0} \mathbb{C} [U \cap \mu^{-1}(\lambda)^{\theta'}]^{k\theta'} \longrightarrow \text{Spec } \mathbb{C} [U \cap \mu^{-1}(\lambda)^\theta]^{G(\alpha)}.$$

It is clear that this morphism is Poisson. □

It follows from the proof of Lemma 2.3 that if $\theta'' \geq \theta' \geq \theta$ then the projective morphism $\mathfrak{M}_\lambda(\alpha, \theta'') \rightarrow \mathfrak{M}_\lambda(\alpha, \theta)$ factors through $\mathfrak{M}_\lambda(\alpha, \theta')$.

2.5. A stratification. Let $x \in \mathfrak{M}_\lambda(\alpha, \theta)$ be a geometric point. We denote by the same symbol a point in the unique closed $G(\alpha)$ -orbit in $\mu^{-1}(\lambda)^\theta$ that maps to x . Then x decomposes into a direct sum $x_1^{e_1} \oplus \cdots \oplus x_k^{e_k}$ of θ -stable representations, with multiplicity. Let $\beta^{(i)} = \dim x_i$. The point x is said to have *representation type* $\tau = (e_1, \beta^{(1)}; \dots; e_k, \beta^{(k)})$. Associated to $x \in \mathfrak{M}_\lambda(\alpha, \theta)_\tau$ is the stabilizer G_τ in $G(\alpha)$ of the lift of x in $\mu^{-1}(\lambda)^\theta$. Even though $\mu^{-1}(\lambda)^\theta$ is not generally affine, the fact that a non-zero morphism between θ -stable representations is an isomorphism implies

Lemma 2.4. *The group G_τ is reductive.*

In fact, it is isomorphic to $\prod_{i=1}^k GL_{e_i}(\mathbb{C})$. Up to conjugation in $G(\alpha)$, it is independent of the lift x . We denote the conjugacy class of a closed subgroup H of $G(\alpha)$ by (H) . Given a reductive subgroup H of $G(\alpha)$, let $\mathfrak{M}_\lambda(\alpha, \theta)_{(H)}$ denote the set of points x such that the stabilizer of x belongs to (H) . We order the conjugacy classes of reductive subgroups of $G(\alpha)$ by $(H) \leq (L)$ if and only if L is conjugate to a subgroup of H . The following result is well-known; see [32, Section 4.5] and the references therein.

Proposition 2.5. *The strata $\mathfrak{M}_\lambda(\alpha, \theta)_\tau = \mathfrak{M}_\lambda(\alpha, \theta)_{(G_\tau)}$ define a finite stratification of $\mathfrak{M}_\lambda(\alpha, \theta)$ into locally closed subsets such that*

$$\mathfrak{M}_\lambda(\alpha, \theta)_{(H)} \subset \overline{\mathfrak{M}_\lambda(\alpha, \theta)_{(L)}} \quad \Leftrightarrow \quad (H) \leq (L).$$

Moreover, each stratum is a symplectic leaf with respect to the natural Poisson bracket on $\mathfrak{M}_\lambda(\alpha, \theta)$.

3. CANONICAL DECOMPOSITIONS OF THE QUIVER VARIETY

In this section we recall the canonical decomposition of quiver varieties described in [7], and show that it holds in slightly greater generality than stated in *loc. cit.*

3.1. Étale local structure. In this section, we recall the étale local structure of $\mathfrak{M}_\lambda(\alpha, \theta)$, as described in [8, Section 4]. Since it is assumed in *loc. cit.* that $\theta = 0$, we provide some details to ensure the results are still applicable in this more general setting. Let $x \in \mathfrak{M}_\lambda(\alpha, \theta)$ be a geometric point and denote by the same symbol a point in the unique closed $G(\alpha)$ -orbit in $\mu^{-1}(\lambda)^\theta$ that maps to x . Then x decomposes into a direct sum $x_1^{e_1} \oplus \cdots \oplus x_k^{e_k}$ of θ -stable representations, with multiplicity. Let $\beta^{(i)} = \dim x_i$. Let Q' be the quiver with k vertices whose double has $2p(\beta^{(i)})$ loops at vertex i and $-(\beta^{(i)}, \beta^{(j)})$ arrows between vertex i and j . The stabilizer of x in $G(\alpha)$ is denoted G_x . The k -tuple $\mathbf{e} = (e_1, \dots, e_k)$ defines a dimension vector for the quiver Q' .

Lemma 3.1. *The closed subgroup G_x of $G(\alpha)$ is isomorphic to $G(\mathbf{e})$ and $\theta|_{G_x}$ is the trivial character.*

Proof. The isomorphism $G_x \simeq G(\mathbf{e})$ follows from the fact that $\text{Hom}_{\Pi^\lambda(Q)}(M_1, M_2) = 0$ if M_1 and M_2 are non-isomorphic θ -stable representations and $\text{End}_{\Pi^\lambda(Q)}(M_i, M_i) = \mathbb{C}$. Under this identification,

$$\theta|_{G(\mathbf{e})} = (\theta \cdot \beta^{(1)}, \dots, \theta \cdot \beta^{(k)}) = (0, \dots, 0) \in \mathbb{Q}^{Q'_0}$$

is the trivial stability condition. □

If X and Y are two G -varieties, then we say that there is a G -equivariant étale isomorphism between a neighborhood of $x \in X$ and $y \in Y$ if there exists a G -variety Z and equivariant morphisms $Y \xleftarrow{\psi} Z \xrightarrow{\phi} X$ and $z \in Z$ such that $\phi(z) = x$, $\psi(z) = y$ and both ϕ and ψ are étale at z . The proof of the following theorem is identical to the proof of [8, Theorem 4.9], it only requires that

one check that the arguments of section 4 of *loc. cit.* are applicable in this slightly more general setting.

Theorem 3.2. *There is a $G(\alpha)$ -equivariant étale isomorphism between a neighborhood of $(1, 0)$ in $G(\alpha) \times_{G_x} \mu_{Q'}^{-1}(0)$ and $x \in \mu^{-1}(\lambda)^\theta$. This induces an étale isomorphism between a neighborhood of 0 in $\mu_{Q'}^{-1}(0)//G(\mathbf{e})$ and the image of x in $\mathfrak{M}_\lambda(\alpha, \theta)$.*

Proof. In the arguments of section 4 of [8], there are two places where one has to be careful since $G(\alpha) \cdot x$ is only assumed to be closed in $\mu^{-1}(\lambda)^\theta$ and not necessarily in $\mu^{-1}(\lambda)$. Firstly, Lemma 2.4 implies that G_x is reductive, therefore [8, Lemma 4.2] is valid in this setting. Let $\mathfrak{g}_x \subset \mathfrak{g}(\alpha)$ be the Lie algebra of G_x and choose a G_x -stable complement L to \mathfrak{g}_x in $\mathfrak{g}(\alpha)$. By [8, Corollary 2.3], we can choose a G_x -stable coisotropic complement C to the isotropic subspace $[\mathfrak{g}(\alpha), x]$ of $\text{Rep}(\overline{Q}, \alpha)$. Define $\nu : C \rightarrow L^*$ by $\nu(c)(l) = \omega(c, lx) + \omega(c, lc) + \omega(c, lx)$ and let μ_x be the composite of μ with the restriction map $\mathfrak{g}^* \rightarrow \mathfrak{g}_x^*$.

Then, in our setting the analogue of [8, Lemma 4.4] states that there exists a G_x -stable affine open neighborhood U of 0 in C such that the assignment $c \mapsto c + x$ induces a G_x -equivariant map $U \cap \mu_x^{-1}(0) \cap \nu^{-1}(0) \rightarrow \mu^{-1}(\lambda)^\theta$, and the induced map

$$(U \cap \mu_x^{-1}(0) \cap \nu^{-1}(0))//G(\mathbf{e}) \longrightarrow \mu^{-1}(\lambda)^\theta//G(\alpha)$$

is étale at 0. It is crucial here that U be affine so that Luna's Fundamental Lemma is applicable. To show that the statement holds, first choose $n \gg 0$ and an $n\theta$ -semi-invariant f such that $f(x) \neq 0$. Define $h : C \rightarrow \mathbb{C}$ by $c \mapsto f(c + x)$. Then we take U to be the affine open subset on which h does not vanish. Since $h(0) = f(x)$, $0 \in U$. Lemma 3.1 implies that h is G_x -invariant, therefore U is G_x -stable. With this in mind, the arguments given in the proof of [8, Lemma 4.4] show that the above stated analogue of that result holds. The remainder of the proof of the theorem follows the proof of [8, Theorem 4.9] *verbatim*. \square

By taking the completion $\widehat{\mathfrak{M}}_\lambda(\alpha, \theta)_x$ of $\mathfrak{M}_\lambda(\alpha, \theta)$ at x and the completion $\widehat{\mathfrak{M}}_0(\mathbf{e}, 0)_0$ of $\mathfrak{M}_0(\mathbf{e}, 0)$ at 0, the formal analogue of Theorem 3.2 states

Corollary 3.3. *There is an isomorphism of formal schemes $\widehat{\mathfrak{M}}_\lambda(\alpha, \theta)_x \simeq \widehat{\mathfrak{M}}_0(\mathbf{e}, 0)_0$.*

Remark 3.4. An easy calculation shows that $p(\alpha) = p(\mathbf{e})$. It can also be deduced from the fact that $\dim \widehat{\mathfrak{M}}_\lambda(\alpha, \theta)_x = \dim \widehat{\mathfrak{M}}_0(\mathbf{e}, 0)_0$. This fact will be useful later.

Remark 3.5. Using Corollary 3.3, if $\alpha = n\beta$ for β indivisible, then at any point of the stratum of $\mathfrak{M}_\lambda(\alpha, \theta)$ corresponding to representations of the form $Y^{\oplus n}$ for $\dim Y = \beta$, the formal neighborhood is isomorphic to the formal neighborhood of the origin of the variety $\mathfrak{M}_0((n), 0)$ for the quiver with one vertex and $p(\beta)$ arrows. Therefore, if $(n, p(\beta)) \neq (2, 2)$ and $p(\beta) \geq 2$, we conclude from [26, Theorem B] that $\mathfrak{M}_0((n), 0)$ does not admit a symplectic resolution, and since it is a cone, the statement is equivalent to the same about the formal neighborhood of zero (cf. Lemma 6.15

below). Therefore a formal neighborhood of any point in this stratum does not admit a symplectic resolution, so neither can $\mathfrak{M}_\lambda(\alpha, \theta)$. This gives an alternative proof of the non-existence portion of Theorem 1.4.

3.2. Hyperkähler twisting. Let $\alpha = m_1\nu^{(1)} + \cdots + m_t\nu^{(t)}$ be the canonical decomposition of α with respect to Σ_λ . It is shown in [7] that

Theorem 3.6. *There is an isomorphism of varieties $\prod_i S^{m_i}(\mathfrak{M}_\lambda(\nu^{(i)}, 0)) \simeq \mathfrak{M}_\lambda(\alpha, 0)$.*

Moreover, if $\nu^{(i)}$ is real then $S^{m_i}(\mathfrak{M}_{\nu^{(i)}}(\lambda, 0)) = \{\text{pt}\}$ and if $\nu^{(i)}$ is non-isotropic imaginary then $m_i = 1$.

Theorem 3.7. *Let $\alpha = n_1\sigma^{(1)} + \cdots + n_k\sigma^{(k)}$ be the canonical decomposition of α with respect to $\Sigma_{\lambda, \theta}$. Then, there is an isomorphism of **Poisson** varieties*

$$\phi : \prod_i S^{n_i}(\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)) \xrightarrow{\sim} \mathfrak{M}_\lambda(\alpha, \theta).$$

The proof of Theorem 3.7 is given at the end of section 3.3. In order to deduce Theorem 3.7 from [7, Theorem 1.1], we use hyperkähler twists. By our main assumption (1), $\lambda \in \mathbb{R}^{Q_0}$.

Proposition 3.8. *Let $\nu = -\lambda - \mathbf{i}\theta$ and consider $\mathfrak{M}_\lambda(\alpha, \theta)$, $\mathfrak{M}_\nu(\alpha, 0)$ as complex analytic spaces. Hyperkähler twisting defines a homeomorphism of stratified spaces*

$$\Psi : \mathfrak{M}_\lambda(\alpha, \theta) \xrightarrow{\sim} \mathfrak{M}_\nu(\alpha, 0),$$

i.e. Ψ restricts to a homeomorphism $\mathfrak{M}_\lambda(\alpha, \theta)_{(H)} \xrightarrow{\sim} \mathfrak{M}_\nu(\alpha, 0)_{(H)}$ for all classes (H) . In particular, the homeomorphism maps stable representations to stable (= simple) representations.

Proof. We follow the setup described in the proof of [5, Lemma 3]. We have moment maps

$$\mu_{\mathbb{C}}(x) = \sum_{a \in Q_1} [x_a, x_{a^*}], \quad \mu_{\mathbb{R}}(x) = \frac{\sqrt{-1}}{2} \sum_{a \in Q_1} [x_a, x_a^\dagger] + [x_{a^*}, x_{a^*}^\dagger].$$

As shown in [27, Corollary 6.2], the Kempf-Ness Theorem says that the embedding $\mu_{\mathbb{C}}^{-1}(\lambda) \cap \mu_{\mathbb{R}}^{-1}(\mathbf{i}\theta) \hookrightarrow \mu_{\mathbb{C}}^{-1}(\lambda)$ induces a bijection

$$\mu_{\mathbb{C}}^{-1}(\lambda) \cap \mu_{\mathbb{R}}^{-1}(\mathbf{i}\theta)/U(\alpha) \xrightarrow{\sim} \mathfrak{M}_\lambda(\alpha, \theta). \quad (3)$$

Since the embedding is clearly continuous and the topology on the quotients $\mu_{\mathbb{C}}^{-1}(\lambda) \cap \mu_{\mathbb{R}}^{-1}(\mathbf{i}\theta)/U(\alpha)$ and $\mathfrak{M}_\lambda(\alpha, \theta)$ is the quotient topology (for the latter space, see [40, Corollary 1.6 and Remark 1.7]), the bijection (3) is continuous.

Define a stratification $\mu_{\mathbb{C}}^{-1}(\lambda) \cap \mu_{\mathbb{R}}^{-1}(\mathbf{i}\theta)/U(\alpha)$ analogous to the stratification of $\mathfrak{M}_\lambda(\alpha, \theta)$ described in section 2.5. Let $\bar{x} \in \mathfrak{M}_\lambda(\alpha, \theta)$ have a θ -polystable lift $x = x_1^{e_1} \oplus \cdots \oplus x_k^{e_k}$ in $\mu_{\mathbb{C}}^{-1}(\lambda) \cap \mu_{\mathbb{R}}^{-1}(\mathbf{i}\theta)$. Then Lemma 3.1 says that $G_x = G(\mathbf{e})$ and [27, Proposition 6.5] implies that $U(\alpha)_x = U(\mathbf{e})$. Hence $G(\alpha)_x = U(\alpha)_x^{\mathbb{C}}$. Therefore the homeomorphism (3) restricts to a bijection

$$(\mu_{\mathbb{C}}^{-1}(\lambda) \cap \mu_{\mathbb{R}}^{-1}(\mathbf{i}\theta)/U(\alpha))_{(K)} \rightarrow \mathfrak{M}_\lambda(\alpha, \theta)_{(K^{\mathbb{C}})}$$

for each (K) .

Let the quaternions $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}\mathbf{i} \oplus \mathbb{R}\mathbf{j} \oplus \mathbb{R}\mathbf{k}$ act on $\text{Rep}(\overline{Q}, \alpha)$ by extending the usual complex structure so that $\mathbf{j} \cdot (x_a, x_{a^*}) = (-x_{a^*}^\dagger, x_a^\dagger)$. In general,

$$(z_1 + z_2\mathbf{j}) \cdot (x_a, x_{a^*}) = (z_1x_a - z_2x_{a^*}^\dagger, z_1x_{a^*} + z_2x_a^\dagger).$$

This action commutes with the action of $U(\alpha)$ and satisfies

$$\mu_{\mathbb{R}}(z \cdot x) = (\|z_1\|^2 - \|z_2\|^2)\mu_{\mathbb{R}}(x) - \mathbf{i}z_1\overline{z_2}\mu_{\mathbb{C}}(x) - \mathbf{i}z_2\overline{z_1}\mu_{\mathbb{C}}(x)^\dagger, \quad (4)$$

$$\mu_{\mathbb{C}}(z \cdot x) = z_1^2\mu_{\mathbb{C}}(x) - z_2^2\mu_{\mathbb{C}}(x)^\dagger - 2\mathbf{i}z_1z_2\mu_{\mathbb{R}}(x), \quad \forall z \in \mathbb{H}. \quad (5)$$

Let $h = (\mathbf{i} - \mathbf{j})/\sqrt{2}$. Then multiplication by h defines a homeomorphism

$$\mu_{\mathbb{C}}^{-1}(\lambda) \cap \mu_{\mathbb{R}}^{-1}(\mathbf{i}\theta) \xrightarrow{\sim} \mu_{\mathbb{C}}^{-1}(-\lambda - \mathbf{i}\theta) \cap \mu_{\mathbb{R}}^{-1}(0)$$

Since multiplication by h commutes with the action of $U(\alpha)$, this homeomorphism descends to a homomorphism

$$(\mu_{\mathbb{C}}^{-1}(\lambda) \cap \mu_{\mathbb{R}}^{-1}(\mathbf{i}\theta)) / U(\alpha) \xrightarrow{\sim} (\mu_{\mathbb{C}}^{-1}(-\lambda - \mathbf{i}\theta) \cap \mu_{\mathbb{R}}^{-1}(0)) / U(\alpha)$$

which preserves the stratification by stabilizer type.

Thus, the map Ψ is the composition of three homeomorphisms, each of which preserves the stratification. \square

Remark 3.9. Our general assumption that $\lambda \in \mathbb{R}^{Q_0}$ if $\theta \neq 0$ is required in the proof of Proposition 3.8 to ensure that multiplication by h lands in $\mu_{\mathbb{R}}^{-1}(0)$. Equation (4) implies that it would suffice to assume more generally that there exists $z \in \mathbb{C}$ such that $|z| = 1$ and $z\lambda \in \mathbb{R}^{Q_0}$. It is natural to expect that Theorem 3.7 holds with out the assumption $\lambda \in \mathbb{R}^{Q_0}$.

Remark 3.10. Using the notion of smooth structures on stratified symplectic spaces, as defined in [41], one can presumably strengthen Proposition 3.8 to the statement that there is a diffeomorphism of stratified symplectic spaces $\mathfrak{M}_\lambda(\alpha, \theta) \xrightarrow{\sim} \mathfrak{M}_\nu(\alpha, 0)$.

Proposition 3.11. *The variety $\mathfrak{M}_\lambda(\alpha, \theta)$ is irreducible and normal.*

Proof. We begin by showing that the variety $\mathfrak{M}_\lambda(\alpha, \theta)$ is connected. Proposition 3.8 implies that $\mathfrak{M}_\lambda(\alpha, \theta)$ is connected if and only if $\mathfrak{M}_\nu(\alpha, 0)$ is connected. The latter is known to be connected by [7, Corollary 1.4].

Next, we show that $\mathfrak{M}_\lambda(\alpha, \theta)$ is irreducible. Since $\mathfrak{M}_\lambda(\alpha, \theta)$ is connected, it suffices to show that, for each \mathbb{C} -point $x \in \mathfrak{M}_\lambda(\alpha, \theta)$, the local ring $\mathcal{O}_{\mathfrak{M}_\lambda(\alpha, \theta), x}$ is a domain. This ring embeds into the formal neighborhood of x in $\mathfrak{M}_\lambda(\alpha, \theta)$. By Corollary 3.3, the formal neighborhood of x in $\mathfrak{M}_\lambda(\alpha, \theta)$ is isomorphic to the formal neighborhood of 0 in $\mathfrak{M}_0(e, 0)$. By [7, Corollary 1.4], this is a domain. Finally, normality is an étalé local property, [33, Remark 2.24 and Proposition 3.17]. Therefore, as in the previous paragraph this follows from Theorem 3.2 and [8, Theorem 1.1]. \square

3.3. The proof of Theorem 3.7. Recall that $\alpha = n_1\sigma^{(1)} + \dots + n_k\sigma^{(k)}$ is the canonical decomposition of α in $R_{\lambda,\theta}^+$. The map ϕ is defined as follows. Let $H(\alpha)$ be the product $G(\sigma^{(1)})^{n_1} \times \dots \times G(\sigma^{(k)})^{n_k}$, thought of as a subgroup of $G(\alpha)$. There is a natural $H(\alpha)$ -equivariant inclusion $\prod_i T^* \text{Rep}(Q, \sigma^{(i)})^{n_i} \hookrightarrow T^* \text{Rep}(Q, \alpha)$. This is an inclusion of symplectic vector spaces. Since the moment map for the action of $H(\alpha)$ on $T^* \text{Rep}(Q, \alpha)$ is the composition of the moment map for $G(\alpha)$ followed by projection from the Lie algebra of $G(\alpha)$ to the Lie algebra of $H(\alpha)$, the above inclusion restricts to an inclusion $\prod_i (\mu_{\sigma^{(i)}}^{-1}(\lambda)^\theta)^{n_i} \hookrightarrow \mu_\alpha^{-1}(\lambda)^\theta$, inducing a map of GIT quotients

$$\prod_i \mathfrak{M}_\lambda(\sigma^{(i)}, \theta)^{n_i} \rightarrow \mathfrak{M}_\lambda(\alpha, \theta).$$

This map, which sends a tuple of representations $(M_{i,j})$ to the direct sum $\bigoplus_{i,j} M_{i,j}$ clearly factors through $\prod_i S^{n_i}(\mathfrak{M}_\lambda(\sigma^{(i)}, \theta))$. It is this map that we call ϕ .

Passing to the analytic topology, Proposition 3.8 implies that we get a commutative diagram

$$\begin{array}{ccc} \prod_i S^{n_i}(\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)) & \longrightarrow & \mathfrak{M}_\lambda(\alpha, \theta) \\ \downarrow & & \downarrow \\ \prod_i S^{n_i}(\mathfrak{M}_\lambda(-\sigma^{(i)} - \mathbf{i}\theta, 0)) & \longrightarrow & \mathfrak{M}_{-\lambda - \mathbf{i}\theta}(\alpha, 0). \end{array} \quad (6)$$

where both vertical arrows are homeomorphisms and the bottom horizontal arrow is an isomorphism by Theorem 3.6. Therefore, we conclude that ϕ is bijective. Since we are working over the complex numbers, and we have shown in Proposition 3.11 that $\mathfrak{M}_\lambda(\alpha, \theta)$ is normal, we conclude by Zariski's main theorem that ϕ is an isomorphism.

Corollary 3.12. *The variety $\mathfrak{M}_\lambda(\alpha, \theta)$ has dimension $2 \sum_{i=1}^k n_i p(\sigma^{(i)})$.*

Proof. By Theorem 3.7, it suffices to show that $\dim \mathfrak{M}_\lambda(\alpha, \theta) = 2p(\alpha)$ if $\alpha \in \Sigma_{\lambda,\theta}$. We note that Proposition 3.8, together with the results of [6], imply that there exists a θ -stable representation of $\Pi^\lambda(Q)$ of dimension α if and only if $\alpha \in \Sigma_{\lambda,\theta}$. Let U be the subset of $\mathfrak{M}_\lambda(\alpha, \theta)$ consisting of θ -stable representations. Since α is assumed to be in $\Sigma_{\lambda,\theta}$, Proposition 3.11 implies that U is a dense open subset of $\mathfrak{M}_\lambda(\alpha, \theta)$. Let V be the open subset of $\text{Rep}(\overline{Q}, \alpha)$ consisting of θ -stable representations. Then U is the image of $\mu^{-1}(\lambda) \cap V$ under the quotient map and hence V is non-empty. The group $G(\alpha)/\mathbb{C}^\times$ acts freely on V and μ is smooth when restricted to V . Thus,

$$\dim U = \dim \text{Rep}(\overline{Q}, \alpha) - 2(\dim G(\alpha) - 1) = 2p(\alpha),$$

as required. □

Finally, we need to check that the morphism ϕ is Poisson. Since both varieties are normal by Proposition 3.11, it suffices to show that ϕ induces an isomorphism of symplectic manifolds between the open leaf of $\mathfrak{M}_\lambda(\alpha, \theta)$ and the open leaf of $\prod_i S^{n_i}(\mathfrak{M}_\lambda(\sigma^{(i)}, \theta))$. By Proposition 2.5, the symplectic leaves of $\mathfrak{M}_\lambda(\alpha, \theta)$ are the strata given by stabilizer type. Therefore the explicit

description of ϕ given at the start of this section shows that ϕ restricts to an isomorphism between strata. In particular, ϕ restricts to an isomorphism between the open leaves.

The symplectic structure on the open leaf of $\mathfrak{M}_\lambda(\alpha, \theta)$ comes from the symplectic structure on $T^*\text{Rep}(Q, \alpha)$. More specifically, the non-degenerate closed form on the latter space restricts to a degenerate $G(\alpha)$ -equivariant two-form on $\mu^{-1}(\lambda)^\theta$. Hence it descends to a closed two-form on $\mathfrak{M}_\lambda(\alpha, \theta)$. The restriction of this two-form to the open leaf is non-degenerate. The two-form on the open leaf of $\prod_i S^{n_i}(\mathfrak{M}_\lambda(\sigma^{(i)}, \theta))$ is defined similarly. Now the point is that under the embedding $\prod_i (\mu_{\sigma^{(i)}}^{-1}(\lambda)^\theta)^{n_i} \hookrightarrow \mu_\alpha^{-1}(\lambda)^\theta$, the $H(\alpha)$ -equivariant closed two-form on $\prod_i (\mu_{\sigma^{(i)}}^{-1}(\lambda)^\theta)^{n_i}$ is simply the pull-back of the $G(\alpha)$ -equivariant closed two-form on $\mu_\alpha^{-1}(\lambda)^\theta$. This implies that the two-form on the open leaf of $\prod_i S^{n_i}(\mathfrak{M}_\lambda(\sigma^{(i)}, \theta))$ is the pull-back, under ϕ , of the symplectic two-form on the open leaf of $\mathfrak{M}_\lambda(\alpha, \theta)$.

4. SMOOTH V.S. STABLE POINTS

As usual, choose deformation parameter $\lambda \in \mathbb{R}^{Q_0}$, stability parameter $\theta \in \mathbb{Q}^{Q_0}$ and dimension vector $\alpha \in \mathbb{N}R_{\lambda, \theta}^+$. The main goal of this section is to prove Theorem 1.13, which says that $x \in \mathfrak{M}_\lambda(\alpha, \theta)$ is θ -canonically stable if and only if it is in the smooth locus of $\mathfrak{M}_\lambda(\alpha, \theta)$.

4.1. The proof of Theorem 1.13. The proof of Theorem 1.13 follows closely the arguments given in [29, Theorem 3.2]. We provide the necessary details that show that the arguments of *loc. cit.* are valid in our setting. First, notice that, under the isomorphism of Theorem 3.7 (2), the open subset of θ -canonically stable points in $\mathfrak{M}_\lambda(\alpha, \theta)$ is the product of the θ -canonically stable points in the spaces $S^{n_i}\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$. Therefore it suffices to show that the set of θ -canonically stable points in $S^{n_i}\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ is precisely the smooth locus. If $\sigma^{(i)}$ is real then $S^{n_i}\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ is a point. If $\sigma^{(i)}$ is an isotropic imaginary root then $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ is a partial resolution of a Kleinian singularity. In particular, it is a 2-dimensional quasi-projective variety. This implies that the smooth locus of $S^{n_i}\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ equals

$$S^{n_i, \circ} \mathfrak{M}_\lambda(\sigma^{(i)}, \theta)_{\text{sm}} := \left\{ \sum_{j=1}^{n_i} p_j \mid p_j \in \mathfrak{M}_\lambda(\sigma^{(i)}, \theta)_{\text{sm}}, p_j \neq p_k \text{ for } j \neq k \right\}.$$

On the other hand, the set of θ -canonically stable points in $S^{n_i}\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ equals $S^{n_i, \circ}U$, where $U \subset \mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ is the set of θ -canonically stable points. Therefore, in this case it suffices to show that $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)_{\text{sm}}$ equals U . Finally, in the case where $\sigma^{(i)}$ is a non-isotropic imaginary root, $n_i = 1$.

Thus, we are reduced to considering the situation where $\alpha \in \Sigma_{\lambda, \theta}$ is an imaginary root. In this case, a point x is θ -canonically stable if and only if it is θ -stable. As in the proof of Corollary 3.12, it is clear from the definition of $\mathfrak{M}_\lambda(\alpha, \theta)$ that the set of θ -stable points is contained in the smooth locus. Therefore it suffices to show that if x is not θ -stable then it is a singular point. As in section 3.1, decompose x into a direct sum $x_1^{e_1} \oplus \cdots \oplus x_\ell^{e_\ell}$ of θ -stable representations with multiplicity. Let

$\beta^{(i)} = \dim x_i$. Let Q' be the quiver with ℓ vertices whose double has $2p(\beta^{(i)})$ loops at vertex i and $-(\beta^{(i)}, \beta^{(j)})$ arrows between vertex i and j . The ℓ -tuple $\mathbf{e} = (e_1, \dots, e_\ell)$ defines a dimension vector for the quiver Q' . By Theorem 3.2, it suffices to show that 0 is contained in the singular locus of $\mathfrak{M}_0(\mathbf{e}, 0)$.

In order to proceed, we require [28, Proposition 1.1], stated in our generality. The proof is identical to the proof given in *loc. cit.* this time using Theorem 3.2.

Proposition 4.1. *Assume that $\alpha \in \Sigma_{\lambda, \theta}$ and let x be a geometric point of $\mathfrak{M}_\lambda(\alpha, \theta)$, of representation type $\tau = (e_1, \beta_1; \dots; e_k, \beta_k)$. Then \mathbf{e} is the dimension vector of a simple $\Pi^0(Q')$ -module i.e. $\mathbf{e} \in \Sigma_0(Q')$.*

Returning to the proof of Theorem 1.13, with Proposition 4.1 in hand, the argument given in the proof of [29, Theorem 3.2] goes through *verbatim*. This completes the proof of Theorem 1.13.

4.2. The proof of Corollary 1.15. By Theorem 1.13, $\mathfrak{M}_\lambda(\alpha, \theta)$ is smooth if and only if every point is θ -canonically stable. As in the reduction argument given at the start of the proof of Theorem 1.13, this means that n_i must be 1 when $\sigma^{(i)}$ is an isotropic imaginary root. Moreover, it is clear that $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ consists only of θ -stable points if and only if $\sigma^{(i)}$ is minimal.

5. THE VARIETY $\mathfrak{X}(n, d)$

Recall that $\mathfrak{X}(n, d)$ denotes the quiver variety

$$\left\{ (X_1, Y_1, \dots, X_d, Y_d) \in \text{End}_{\mathbb{C}}(\mathbb{C}^n) \mid \sum_{i=1}^d [X_i, Y_i] = 0 \right\} // \text{GL}(n, \mathbb{C}).$$

In this section we recall results of Lehn-Kaledin [25] and Lehn-Kaledin-Sorger [26], which say when $\mathfrak{X}(n, d)$ admits a projective symplectic resolution. We note that $\mathfrak{X}(n, d)$ is an irreducible, normal affine variety of dimension $2(n^2(d-1) + 1)$.

5.1. The case $(n, d) = (2, 2)$. Let $W = \mathfrak{sl}_2$ and (V, ω) a 4-dimensional symplectic vector space. Let κ denote the Killing form on W . Then $\kappa \otimes \omega$ is a symplectic form on $W \otimes V$. We identify $\mathfrak{sp}(V)^*$ with $\mathfrak{sp}(V)$ via its Killing form. There is an action of $\text{PGL}(2)$ on W by conjugation and hence on $W \otimes V$. This action is Hamiltonian and commutes with the natural action of $\text{Sp}(V)$ on $W \otimes V$. The moment map for the action of $\text{PGL}(2)$ is given by

$$\begin{aligned} \mu \left(\sum_i A_i \otimes v_i \right) &= \sum_{i,j} A_i A_j \omega(v_i, v_j) \\ &= \sum_{i < j} [A_i, A_j] \omega(v_i, v_j). \end{aligned}$$

The moment map for the action of $\mathrm{Sp}(V)$ is given by $\sum_i A_i \otimes v_i \mapsto \nu(\sum_i A_i \otimes v_i)$, where

$$\nu \left(\sum_i A_i \otimes v_i \right) (u) = \sum_{i,j} \kappa(A_i, A_j) \omega(v_i, u) v_j.$$

Since the actions of $\mathrm{PGL}(2)$ and $\mathrm{Sp}(V)$ on $\mu^{-1}(0)$ commute, the map ν descends to a map $\mu^{-1}(0)/\mathrm{PGL}(2) \rightarrow \mathfrak{sp}(V)$, which we also denote by ν . Let $\mathcal{N}_2^2 \subset \mathfrak{sp}(V)$ be the set $\{B \mid B^2 = 0, \mathrm{rk} B = 2\}$. The set \mathcal{N}_2^2 is a 6-dimensional adjoint $\mathrm{Sp}(V)$ -orbit. Its closure $\mathcal{N} := \overline{\mathcal{N}_2^2} = \mathcal{N}_2^2 \cup \mathcal{N}_1^2 \cup \{0\}$ consists of three $\mathrm{Sp}(V)$ -orbits and one can check that $\overline{\mathcal{N}_1^2} \simeq \mathbb{C}^4/\mathbb{Z}_2$, where \mathbb{Z}_2 acts on \mathbb{C}^4 with weights $(-1, -1, -1, -1)$. The following result is proven in [25].

Theorem 5.1. *The map ν defines an isomorphism $\mu^{-1}(0)/\mathrm{PGL}(2) \xrightarrow{\sim} \mathcal{N}$ of Poisson varieties. In particular, $\mu^{-1}(0)/\mathrm{PGL}(2)$ is a symplectic singularity.*

Taking trace of the matrices $(X_1, X_2, Y_1, Y_2) \in \mathfrak{X}(2, 2)$ defines an isomorphism of symplectic singularities $\mathfrak{X}(2, 2) \simeq \mu^{-1}(0)/\mathrm{PGL}(2) \times \mathbb{A}^4$, where \mathbb{A}^4 is given the usual symplectic structure. Thus, $\mathfrak{X}(2, 2) \simeq \mathcal{N} \times \mathbb{A}^4$.

Remark 5.2. Let W be an arbitrary vector space equipped with a symmetric non-degenerate form. Then $\mathrm{SO}(W)$ acts symplectically on $W \otimes V$ and we have a moment map $\mu : W \otimes V \rightarrow \mathfrak{so}(W)^*$. Then it would be interesting to know when μ is flat; does the variety $\mu^{-1}(0)/\mathrm{SO}(W)$ admit a symplectic resolution? And what is the image of $\mu^{-1}(0)/\mathrm{SO}(W)$ in $\mathfrak{sp}(V)$ under the moment map μ ? The same questions makes sense in our quiver setting of $\mathrm{PGL}(n, \mathbb{C})$ acting on $\mathfrak{gl}_n \otimes W$; see [25, Remark 4.5]. It seems the former situation should be easier to understand than the latter.

The following is a consequence of the arguments of [25, Remark 4.6], applied to our situation. We provide details for the reader's benefit.

Corollary 5.3. *Let α be an indivisible, non-isotropic imaginary root with $p(\alpha) = 2$. For all λ, θ such that $\alpha \in \Sigma_{\lambda, \theta}$, the quiver variety $\mathfrak{M}_\lambda(2\alpha, \theta)$ admits a projective symplectic resolution.*

Proof. Since α is non-isotropic imaginary, Lemma 2.2 implies that 2α is also a non-isotropic imaginary root. Choose a generic stability parameter $\theta' \geq \theta$ with $\theta' \cdot \beta \neq 0$ for all non-zero $\beta \leq 2\alpha$, $\beta \neq \alpha$. Then the projective Poisson morphism $\mathfrak{M}_\lambda(2\alpha, \theta') \rightarrow \mathfrak{M}_\lambda(2\alpha, \theta)$ of Lemma 2.3 is a partial projective resolution. The proof of Lemma 6.11 below shows that if $Y \rightarrow \mathfrak{M}_\lambda(2\alpha, \theta')$ is a projective symplectic resolution then so is the composite $Y \rightarrow \mathfrak{M}_\lambda(2\alpha, \theta)$ i.e. it is enough to show that we can resolve $\mathfrak{M}_\lambda(2\alpha, \theta')$ symplectically. Fix $X = \mathfrak{M}_\lambda(2\alpha, \theta')$. Then $X = X_2 \sqcup X_1 \sqcup X_0$, where, by Theorem 1.13, X_0 is the smooth locus consisting of θ' -stable representations, X_1 parameterizes representations $M = M_1 \oplus M_2$ with $\dim M_1 = \dim M_2 = \alpha$, $M_1 \not\cong M_2$ are θ' -stable representations and X_2 consists of all points M^2 , with $\dim M = \alpha$. By Proposition 2.5, X_2 and $X_2 \cup X_1$ are closed in X .

Let \mathcal{J} be the ideal sheaf of X_1 in $X_{\leq 1}$ and $j : X_{\leq 1} \hookrightarrow X$. Since X is normal and X_0 has codimension 6 in X , $j_*\mathcal{J}$ is a sheaf of ideals on X . Let \tilde{X} denote the blowup of X along the sheaf

of ideals $j_*\mathcal{J}^\ell$. The corollary will follow from the following claim: for $\ell \gg 0$, $\tilde{X} \rightarrow X$ is a projective symplectic resolution. Clearly, it is a projective birational morphism, therefore we just need to show that \tilde{X} is smooth and the symplectic 2-form on X_0 extends to a symplectic 2-form on \tilde{X} . We check this in a neighborhood of $x \in X_2$. Replacing X by some affine open neighborhood of x , Theorem 3.2 says that there is an affine Z with

$$\begin{array}{ccc} & Z & \\ \pi \swarrow & & \searrow \rho \\ X & & \mathfrak{X}(2,2) \end{array}$$

where π and ρ are étale. Theorem 3.2 also says that $Z_2 := \pi^{-1}(X_2) = \rho^{-1}(\mathfrak{X}(2,2)_2)$ and $Z_1 := \pi^{-1}(X_1) = \rho^{-1}(\mathfrak{X}(2,2)_1)$. Denote by k and l the embeddings $Z_{\leq 1} \hookrightarrow Z$ and $\mathfrak{X}(2,2)_{\leq 1} \hookrightarrow \mathfrak{X}(2,2)$ respectively. Similarly, \mathcal{K} and \mathcal{L} will denote the sheaf of ideals defining Z_1 in $Z_{\leq 1}$ and $\mathfrak{X}(2,2)_1$ in $\mathfrak{X}(2,2)_{\leq 1}$ respectively. Then flat base change [22, III, Proposition 9.3] implies that

$$k_*\mathcal{K}^\ell = \pi^*j_*\mathcal{J}^\ell = \rho^*l_*\mathcal{L}^\ell.$$

Hence, if $\tilde{Z} \rightarrow Z$ and $\tilde{\mathfrak{X}}(2,2) \rightarrow \mathfrak{X}(2,2)$ denote blowup along $k_*\mathcal{K}^\ell$ and $l_*\mathcal{L}^\ell$ respectively, then

$$\tilde{Z} \simeq \tilde{X} \times_X Z \simeq \tilde{\mathfrak{X}}(2,2) \times_{\mathfrak{X}(2,2)} Z. \quad (7)$$

As noted in [25, Remark 4.6], $\tilde{\mathfrak{X}}(2,2) \rightarrow \mathfrak{X}(2,2)$ is a projective symplectic resolution. Now Lemma 5.4 and (7) imply that $\tilde{X} \rightarrow X$ is a projective symplectic resolution. \square

The following result is standard.

Lemma 5.4. *Let X be a symplectic singularity and $\pi : \tilde{X} \rightarrow X$ a projective morphism. Then π is a projective symplectic resolution if and only if it is so after a surjective étale base change i.e. being a symplectic resolution is an étale local property.*

Notice that we are not making the (false) claim that X admits a symplectic resolution if and only if it does so étale locally.

Proof. Passing to the generic points of \tilde{X} and X , the fact that a surjective étale morphism is faithfully flat implies that π is birational if and only if it is so after base change. Therefore it suffices to check that the extension ω' of the pullback $\pi^*\omega$ is non-degenerate. If $b : Z \rightarrow X$ is a surjective étale morphism, then so too is $\tilde{b} : \tilde{Z} = \tilde{X} \times_X Z \rightarrow \tilde{X}$. The form ω' will be non-degenerate if and only if $\tilde{b}^*\omega'$ is non-degenerate. \square

6. DIVISIBLE NON-ISOTROPIC IMAGINARY ROOTS

In this section, which is the technical heart of the paper, we consider the case of a divisible non-isotropic imaginary root. Fix $\alpha \in \Sigma_{\lambda,\theta}$ be an indivisible non-isotropic imaginary root, and let $n \geq 2$ such that $(p(\alpha), n) \neq (2, 2)$. We prove the key result, Corollary 6.8, which says that if θ is generic then $\mathfrak{M}_\lambda(n\alpha, \theta)$ is a locally factorial variety.

6.1. A *weighted partition* ν of n is a sequence $(\ell_1, \nu_1; \dots; \ell_k, \nu_k)$, where $\nu_1 \geq \nu_2 \geq \dots$ and $\sum_{i=1}^k \ell_i \nu_i = n$. Recall from Proposition 2.5 that the quiver variety $\mathfrak{M}_\lambda(\alpha, \theta)$ has a finite stratification by representation type.

Lemma 6.1. *Let $\alpha \in \Sigma_{\lambda, \theta}$ be an indivisible non-isotropic imaginary root. Let $n \geq 2$ such that $(p(\alpha), n) \neq (2, 2)$. Assume that $\theta \cdot \beta \neq 0$ for all $\beta \leq n\alpha$ not a multiple of α .*

- (1) *The set $\Sigma_{\lambda, \theta}$ contains $\{m\alpha \mid m \geq 1\}$.*
- (2) *The strata $\mathfrak{M}_\lambda(n\alpha, \theta)_\nu$ of $\mathfrak{M}_\lambda(n\alpha, \theta)$ are parameterized by weighted partitions of n and*

$$\dim \mathfrak{M}_\lambda(n\alpha, \theta)_\nu = 2 \left(k + (n-1) \sum_{i=1}^k \nu_i^2 \right).$$

- (3) *Since $(p(\alpha), n) \neq (2, 2)$, we have $\dim \mathfrak{M}_\lambda(n\alpha, \theta) - \dim \mathfrak{M}_\lambda(n\alpha, \theta)_\nu \geq 4$ for all $\nu \neq (1, n)$.*

Proof. The first claim follows from Lemma 2.2.

Set $d := p(\alpha)$. Then $p(n\alpha) = n^2(d-1) + 1$. Our assumptions imply that a stratum of $\mathfrak{M}_\lambda(\alpha, \theta)$ consists of all representations of the form $x = x_1^{\oplus \ell_1} \oplus \dots \oplus x_k^{\oplus \ell_k}$, where the x_i are pairwise non-isomorphic simple modules of dimension $\nu_i \alpha$ and $n = \sum_{i=1}^k \ell_i \nu_i$. Then the dimension formula follows from [6, Theorem 1.3].

Finally, notice that

$$\begin{aligned} \dim \mathfrak{M}_\lambda(n\alpha, \theta) - \dim \mathfrak{M}_\lambda(n\alpha, \theta)_\nu &= 2(n^2(d-1) + 1) - 2 \sum_{i=1}^k (\nu_i^2(d-1) + 1) \\ &= 2(d-1) \sum_{i,j=1}^k (\ell_i \ell_j - \delta_{i,j}) \nu_i \nu_j - 2(k-1). \end{aligned}$$

Since $\sum_{i,j=1}^k (\ell_i \ell_j - \delta_{i,j}) \nu_i \nu_j - (k-1) \geq 1$, we clearly have $\dim \mathfrak{M}_\lambda(n\alpha, \theta) - \dim \mathfrak{M}_\lambda(n\alpha, \theta)_\nu \geq 4$ when $d > 2$. When $d = 2$, a simple computation shows that $\dim \mathfrak{M}_\lambda(n\alpha, \theta) - \dim \mathfrak{M}_\lambda(n\alpha, \theta)_\nu = 2$ if and only if $n = 2$ and $\nu = (1, 1; 1, 1)$. \square

In particular, Lemma 6.1 describes the stratification of $\mathfrak{X}(n, d)$. Since $p(\alpha) > 1$, there exist infinitely many non-isomorphic simple $\Pi^\lambda(Q)$ -modules of dimension α . Therefore, for all representation types $\tau = (\tau_1, n_1\alpha; \dots; \tau_k, n_k\alpha)$ with $\sum_i \tau_i n_i = n$, the stratum $\mathfrak{M}_\lambda(n\alpha, \theta)_\tau$ is non-empty. Let U be the union of all strata of “type τ ”.

Lemma 6.2. *The subset U is open in $\mathfrak{M}_\lambda(n\alpha, \theta)$.*

Proof. Since the stratum of representation type $\rho = (n, \alpha)$ is contained in the closure of all the other strata of type τ , it suffices to show that there is no stratum $\beta = (e_1, \beta^{(1)}; \dots; e_l, \beta^{(l)})$ of any other type such that $\mathfrak{M}_\lambda(n\alpha, \theta)_\rho \subset \overline{\mathfrak{M}_\lambda(n\alpha, \theta)_\beta}$. Assume otherwise. If $G_\rho \simeq GL_n(\mathbb{C})$ is the stabilizer of some $x \in \mathfrak{M}_\lambda(n\alpha, \theta)_\rho$, then there exists some $y \in \mathfrak{M}_\lambda(n\alpha, \theta)_\beta$ whose stabilizer G_β is contained in G_ρ . Let V_i be the $n\alpha_i$ -dimensional vector space at the vertex i on which $G(n\alpha)$ acts. Then, for each $g \in G(n\alpha)$ and $u \in \mathbb{C}^\times$, the u -eigenspace of g is the direct sum over the u -eigenspaces $g|_{V_i}$. In

particular, it has a well-defined dimension vector. Now the elements g of G_ρ all have the property that the dimension vector of the u -eigenspace of g is of the form $r\alpha$ for some $r \in \mathbb{Z}_{\geq 0}$. On the other hand, since β is not “of type τ ”, there is some i such that $e_i\beta^{(i)} \neq r\alpha$ for any r . Take $u \neq 1$ and $g \in G_\beta$ that rescales the summand of y of dimension $e_i\beta^{(i)}$ by u and is the identity on all other summands. Then the u -eigenspace of g has dimension vector $e_i\beta^{(i)}$ which implies that $G_\beta \not\subset G_\rho$ - a contradiction. Thus, U is open. \square

The open subset of $\mu^{-1}(\lambda)$ consisting of points with trivial stabilizer under $PG(n\alpha)$ is denoted $\mu^{-1}(\lambda)_{\text{free}}^\theta$. The image of $\mu^{-1}(\lambda)_{\text{free}}^\theta$ in $\mathfrak{M}_\lambda(n\alpha, \theta)$ is denoted $\mathfrak{M}_\lambda(n\alpha, \theta)_{\text{free}}$. Let $\mu^{-1}(\lambda)_{\text{stable}}^\theta \subset \mu^{-1}(\lambda)^\theta$ denote the open subset of θ -stable representations. Since $\alpha \in \Sigma_{\lambda, \theta}$, the set $\mu^{-1}(\lambda)_{\text{stable}}^\theta$ is non-empty. The image of $\mu^{-1}(\lambda)_{\text{stable}}^\theta$ in $\mathfrak{M}_\lambda(n\alpha, \theta)$ is denoted $\mathfrak{M}_\lambda(n\alpha, \theta)_{\text{stable}}$.

Remark 6.3. For $n \geq 2$, the inclusion $\mu^{-1}(\lambda)_{\text{stable}}^\theta \subset \mu^{-1}(\lambda)_{\text{free}}^\theta$ is proper. This is due to the fact that in this case there exist non θ -stable, but indecomposable, representations M with dimension vector $n\alpha$, which fit into a short exact sequence

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

where M' and M are non-isomorphic, θ -stable representations of dimension $n'\alpha$ and $n''\alpha$ respectively (for some $n' + n'' = n$).

6.2. Local factoriality of $\mathfrak{M}_\lambda(n\alpha, \theta)$. A closed point $x \in X$ is said to be factorial if the local ring $\mathcal{O}_{X, x}$ is a unique factorization domain. We say that X is locally factorial if X is factorial at every closed point. If $\xi : \mu^{-1}(\lambda)^\theta \rightarrow \mathfrak{M}_\lambda(n\alpha, \theta)$, then let $V = \xi^{-1}(U)$, where U is the open subset of Lemma 6.2.

Proposition 6.4. *V is a local complete intersection, locally factorial and normal. Moreover, the complement to $\mu^{-1}(\lambda)_{\text{free}}^\theta$ in V has codimension at least 4.*

Proof. The proposition follows from the various results of [26]. First choose a point $x \in V$ whose orbit in $\mu^{-1}(\lambda)^\theta$ is closed. Then, in the proof of Theorem 3.2, there is constructed a slice S to $G(\alpha)$ -orbit through x ; S is the image of $U \cap \mu_x^{-1}(0) \cap \nu^{-1}(0)$. We wish to show that S is a complete intersection, smooth in codimension 3 (i.e. property (R_3) holds), and normal at x . These are étale local properties, therefore it suffices by Theorem 3.2 to check that they hold for points of $\mu_{Q'}^{-1}(0)$. This is deduced from [26, Proposition 3.6], which says that $\mu_{Q'}^{-1}(0)$ is an irreducible, normal complete intersection, provided the hypotheses of the proposition hold. Moreover, $\mu_{Q'}^{-1}(0)$ satisfies (R_3) in this situation. The representation type of x is $(\tau_1, n_1\alpha; \dots; \tau_k, n_k\alpha)$, for some n_i and τ_j . To see that the hypotheses hold, we note that the vector n of *loc. cit.* is our \mathbf{e} , $d_{i,j} - 2\delta_{i,j}$ of *loc. cit.* equals $-(n_i\alpha, n_j\alpha) = -n_i n_j \langle \alpha, \alpha \rangle = -2n_i n_j \langle \alpha, \alpha \rangle$ and we have assumed that α is non-isotropic imaginary, whence $\langle \alpha, \alpha \rangle < 0$. This implies that the a from *loc. cit.* is at least 2. Thus, [26, Proposition 3.6] applies. Moreover, situations 1. and 2. from [26, Proposition 3.6] only arise when $(n, p(\alpha)) = (2, 2)$, which we have explicitly avoided.

Being a local complete intersection, normal and satisfying (R_3) are open conditions. Therefore there is an open neighborhood of x in S in which they hold. Since the morphism $G(\alpha) \times S \rightarrow V$ is smooth at x , there is an open neighborhood of x in V which is a local complete intersection, normal and satisfies (R_3) . Thus, there is an open neighborhood of every closed orbit in V in which these properties hold. Since $V = \xi^{-1}(U)$, the closure $\overline{\mathcal{O}}$ (in V) of each $G(\alpha)$ -orbit contains a closed (in $\mu^{-1}(\lambda)^\theta$) $G(\alpha)$ -orbit \mathcal{O}' . If B is the open neighborhood of \mathcal{O}' in which the three properties hold then $B \cap \overline{\mathcal{O}}$ is open and dense in $\overline{\mathcal{O}}$. Thus, $B \cap \mathcal{O} \neq \emptyset$. Since $G(\alpha)$ acts transitively on \mathcal{O} , $B \cap \mathcal{O} = \mathcal{O}$ and we deduce that V is a local complete intersection, normal and satisfies (R_3) . It follows from a theorem of Grothendieck, [26, Theorem 3.12], that each point of V is factorial.

The second statement of the proposition can also be deduced from [26, Proposition 3.6]. Though the statement of that proposition does not explicit refer to the locus of $\mu_{Q'}^{-1}(0)$ where $PG(\mathbf{e})$ acts freely, in the proof the author explicitly show that its complement has codimension at least 4. Repeating the argument of the previous paragraph, we deduce that the complement in V of the locus where $PG(\alpha)$ acts freely also has codimension at least 4. \square

Lemma 6.5. *Every $PG(\alpha)$ -equivariant line bundle on $\mu^{-1}(\lambda)_{\text{free}}^\theta$ extends to a $PG(\alpha)$ -equivariant line bundle on V .*

Proof. By Proposition 6.4, the complement to $\mu^{-1}(\lambda)_{\text{free}}^\theta$ in V has codimension at least 4. Therefore, the fact that V is normal and locally factorial implies that

$$\text{Pic}(V) = \text{Div}(V) = \text{Div}\left(\mu^{-1}(\lambda)_{\text{free}}^\theta\right) = \text{Pic}\left(\mu^{-1}(\lambda)_{\text{free}}^\theta\right).$$

Hence if L_0 is a $PG(\alpha)$ -equivariant line bundle on $\mu^{-1}(\lambda)_{\text{free}}^\theta$, forgetting the equivariant structure, there is an extension L to V . To show that the extension L has a $PG(\alpha)$ -equivariant structure, one repeats the argument of [14, Lemme 5.2]. \square

The result that allows us to descend local factoriality from V to the quotient U is the following theorem by Drezet. Since the version given in [13] concerns the moduli space of semi-stable sheaves on a smooth surface, we provide full details to ensure the arguments are applicable in our situation.

Theorem 6.6 ([13], Theorem A). *Let $x \in U$ and y a lift in V . The following are equivalent:*

- (i) *The local ring $\mathcal{O}_{U,x}$ is a unique factorization domain.*
- (ii) *For every line bundle M_0 on $\mathfrak{M}_\lambda(\alpha, \theta)_{\text{free}}$, there exists an open subset $U_0 \subset U$ containing both x and $\mathfrak{M}_\lambda(\alpha, \theta)_{\text{free}}$ such that M_0 extends to a line bundle M on U_0 .*
- (iii) *For every $PG(\alpha)$ -equivariant line bundle L on V , the stabilizer of y acts trivially on the fiber L_y .*

Proof. Recall that $\mathcal{O}_{U,x}$ is a unique factorization domain if and only if every height one prime is principal. Geometrically, this means that for every hypersurface Y of U , the sheaf of ideals \mathcal{I}_Y is free at x .

(i) implies (ii). It suffices to assume that $L_0 = \mathcal{I}_Y$, where Y is a hypersurface in $\mathfrak{M}_\lambda(\alpha, \theta)_{\text{free}}$. If \overline{Y} is the closure of Y in U , then $L = \mathcal{I}_{\overline{Y} \cap U'}$ is the required extension.

(ii) implies (i). Let Y be a hypersurface in U . We wish to show that \mathcal{I}_Y is free at x . Let L be the extension of \mathcal{I}_Y to U' . The line bundle L corresponds to a Cartier divisor D on U' ; $L = \mathcal{O}_{U'}(D)$. Then,

$$\mathcal{I}_Y = \mathcal{O}_{\mathfrak{M}_\lambda(\alpha, \theta)_{\text{free}}}(D \cap \mathfrak{M}_\lambda(\alpha, \theta)_{\text{free}}),$$

and the divisors Y and $-D \cap \mathfrak{M}_\lambda(\alpha, \theta)_{\text{free}}$ are linearly equivalent. Since the codimension of the complement to $\mathfrak{M}_\lambda(\alpha, \theta)_{\text{free}}$ in U has codimension at least two by Lemma 6.1 and U is normal by Proposition 3.11, $\overline{Y} \simeq -D$. Hence $L = \mathcal{I}_{\overline{Y}}$ is free at x .

(ii) implies (iii). Suppose that L is a $PG(\alpha)$ -equivariant line bundle on V . Since $PG(\alpha)$ acts freely on $\mu^{-1}(\lambda)_{\text{free}}^\theta$, the restriction $L|_{\mu^{-1}(\lambda)_{\text{free}}^\theta}$ descends to the line bundle $M_0 = (L|_{\mu^{-1}(\lambda)_{\text{free}}^\theta})/PG(\alpha)$ on $\mathfrak{M}_\lambda(\alpha, \theta)_{\text{free}}$. Let M be the extension of M_0 to U' . Then the $PG(\alpha)$ -equivariant line bundle ξ^*M agrees with L on $\mu^{-1}(\lambda)_{\text{free}}^\theta$. This implies, as in the previous paragraph, that $\xi^*M = L$ on $\xi^{-1}(U')$. In particular, since $y \in \xi^{-1}(U')$, the stabilizer of y acts trivially on L_y .

(iii) implies (ii). Let M_0 be a line bundle on $\mathfrak{M}_\lambda(\alpha, \theta)_{\text{free}}$. By Lemma 6.5, ξ^*M_0 extends to a $PG(\alpha)$ -equivariant line bundle L on V . Recall by definition of lift that $PG(\alpha) \cdot y$ is closed in $\mu^{-1}(\lambda)^\theta$. Therefore Lemma 6.7 below says that there is an affine open neighborhood U' of x such that $PG(\alpha)_{y'}$ acts trivially on $L_{y'}$ for all $y' \in \xi^{-1}(U')$ such that $PG(\alpha) \cdot y'$ is closed in $\mu^{-1}(\lambda)^\theta$. Let $U_0 = U' \cup \mathfrak{M}_\lambda(\alpha, \theta)_{\text{free}}$. Then, by descent [13, Theorem 1.1], there exists a line bundle M on U_0 such that $\xi^*M \simeq L$. In particular, M extends M_0 . \square

Let Y be a variety admitting an algebraic action of a reductive group G . Assume that there exists a good quotient $\xi : Y \rightarrow X = Y//G$. The following result, which says that the descent locus of an equivariant line bundle is open, is presumably well-known, but we were unable to find it in the literature.

Lemma 6.7. *Let L be a G -equivariant line bundle on Y and $y \in Y$ a closed point such that the orbit $\mathcal{O} = G \cdot y$ is closed and the stabilizer G_y of y acts trivially on the fiber L_y . Then there exists an affine open neighborhood U of $\xi(y)$ such that the stabilizer $G_{y'}$ acts trivially on $L_{y'}$ for all $y' \in \xi^{-1}(U)$ such that $G \cdot y'$ is closed.*

Proof. The proof of the lemma can be easily deduced from the proof of [14, Theorem 2.3]. It is shown in *loc. cit.* that one can find a G -invariant section $s' : \mathcal{O} \rightarrow L|_{\mathcal{O}}$, which trivializes $L|_{\mathcal{O}}$. As explained in *loc. cit.*, the fact that \mathcal{O} is closed in Y implies that one can lift s' to a G -invariant section $s \in \Gamma(\xi^{-1}(U'), L)$, where U' is some affine open neighborhood of $\xi(y)$. Let W be the (non-empty) open subset of $\xi^{-1}(U')$ consisting of all points y' such that $s(y') \neq 0$ i.e. s trivializes L over W . Then it suffices to show that there is some affine neighborhood U of $\xi(y)$ such that $\xi^{-1}(U) \subset W$. Again, following *loc. cit.*, the sets $\xi^{-1}(U') \setminus W$ and \mathcal{O} are G -stable closed subsets of $\xi^{-1}(U')$. Therefore the fact that ξ is a good quotient implies that $\xi(\xi^{-1}(U') \setminus W)$ and

$\xi(\mathcal{O}) = \{\xi(y)\}$ are closed, disjoint subsets of U' . Thus, there exists an affine neighborhood U of $\xi(y)$ such that $U \cap \xi(\xi^{-1}(U') \setminus W) = \emptyset$, as required. \square

Corollary 6.8. *The variety U is locally factorial.*

Proof. First recall from the proof of Lemma 6.2, the stratum of type $\rho = (n, \alpha)$ is contained in the closure of all other strata in U . If y is a lift in $\mu^{-1}(\lambda)^\theta$ of a point of $\mathfrak{M}_\lambda(n\alpha, \theta)_\rho$ then y corresponds to a representation $M_0^{\oplus n}$, where $M_0 \in \mathfrak{M}_\lambda(\alpha, \theta)$ is a simple $\Pi^\lambda(Q)$ -module. Therefore $PG(\alpha)_y = PGL_n$ has no non-trivial characters. In particular, $PG(\alpha)_y$ will act trivially on L_y for any $PG(\alpha)$ -equivariant line bundle on V . Hence, we deduce from Theorem 6.6 that $\mathfrak{M}_\lambda(n\alpha, \theta)$ is factorial at every point of $\mathfrak{M}_\lambda(n\alpha, \theta)_\rho$.

Now consider an arbitrary stratum $\mathfrak{M}_\lambda(n\alpha, \theta)_\tau$ in U . If $\mathfrak{M}_\lambda(n\alpha, \theta)$ is factorial at one point of the stratum then it will be factorial at every point in the stratum (for a rigorous proof of this fact, repeat the argument given in the proof of [26, Theorem 5.3]). On the other hand, a deep theorem of Boissière, Gabber and Serman [3] says that the subset of factorial points of U is an open subset. Since this open subset is a union of strata and contains the unique closed stratum, it must be the whole of U . \square

Remark 6.9. Notice that if θ is generic then $U = \mathfrak{M}_\lambda(n\alpha, \theta)$. Hence Corollary 6.8 says that $\mathfrak{M}_\lambda(n\alpha, \theta)$ is a locally factorial variety. This is precisely the statement of Theorem 1.7.

6.3. The proof of Proposition 1.2. By Proposition 3.11, we know that $\mathfrak{M}_\lambda(\alpha, \theta)$ is irreducible and normal. Therefore, it suffices to show that it admits symplectic singularities. Since the isomorphism of Theorem 3.7 is Poisson, it suffices to show that the varieties $S^{n_i} \mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ admit symplectic singularities. If $\sigma^{(i)}$ is real there is nothing to check.

Lemma 6.10. *Let X be a smooth irreducible Poisson variety and Y a symplectic manifold. If $\pi : Y \rightarrow X$ is a birational, surjective Poisson morphism, then it is an isomorphism.*

Proof. Since the morphism is birational, there is a dense open subset $U \subset X$ over which it is an isomorphism. By [22, I, Corollary 6.12], the complement of U has codimension at least two. On the other hand, since X is smooth, the locus where the Poisson structure on X is degenerate has codimension one. Therefore, X is symplectic too. This implies that $d_y \pi$ is an isomorphism for all $y \in Y$. Thus, by Zariski's Main Theorem, π is an isomorphism. \square

Lemma 6.11. *Let X be a normal Poisson variety and assume that $\pi : Y \rightarrow X$ is a projective birational Poisson morphism from a variety Y with symplectic singularities. Then X has symplectic singularities.*

Proof. Let $\rho : Z \rightarrow Y$ be a projective resolution of singularities. If ω' is the symplectic 2-form on the smooth locus of Y then $\rho^* \omega'$ extends to a regular form on Z . Let $Y' = \pi^{-1}(X_{\text{sm}})$. Then, since $\pi : Y' \rightarrow X_{\text{sm}}$ is proper and birational, it is surjective. Lemma 6.10 implies that it is an

isomorphism. In particular, there is a symplectic 2-form on X_{sm} such that the Poisson structure on X_{sm} is non-degenerate and induced from ω . Moreover, $\pi^*\omega = \omega'$. Thus, $(\pi \circ \rho)^*\omega = \rho^*\omega'$ extends to a regular form, and hence X has symplectic singularities. \square

Remark 6.12. One can drop the assumption in Lemma 6.11 that X is Poisson and π is a Poisson morphism; since $R^0\pi_*\mathcal{O}_Y = \mathcal{O}_X$ it naturally inherits a Poisson structure making π Poisson.

If $\sigma^{(i)}$ is an indivisible non-isotropic imaginary root then $n_i = 1$ and choosing a generic stability parameter $\theta' \geq \theta$ defines a projective, Poisson resolution $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)' \rightarrow \mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ with $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)'$ a symplectic manifold; see [8, Section 8]. Similarly, if $\sigma^{(i)}$ is isotropic imaginary then it is well-known that one can frame the quiver so that there exists a projective, Poisson resolution of singularities from a quiver variety that is a symplectic manifold. Thus, Lemma 6.11 implies that $S^{n_i}\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ admits symplectic singularities in these two cases.

Similarly, if $(n, d) := (\gcd(\sigma^{(i)}), p(\gcd(\sigma^{(i)})^{-1}\sigma^{(i)})) = (2, 2)$, then Corollary 5.3 and Lemma 6.11 imply that $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ has symplectic singularities. Therefore, it suffices to show that $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ has symplectic singularities when $(n, d) \neq (2, 2)$. Again, choose a generic stability parameter $\theta' \geq \theta$. Then $\pi : \mathfrak{M}_\lambda(\sigma^{(i)}, \theta') \rightarrow \mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ is projective and Poisson by Lemma 2.3. Moreover, since both $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta')$ and $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ are irreducible by Proposition 3.11, and a generic element of $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ is θ -stable, the map π is birational. Thus, by Lemma 6.11, it suffices to show that $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta')$ admits symplectic singularities. This follows from Flenner's Theorem [16], once we show that the singular locus of $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta')$ has codimension at least four. By Theorem 1.13, the singular locus of $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta')$ is the union of all strata except the open stratum. By Lemma 6.1 each of these strata has codimension at least 4 in $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta')$.

6.4. The proof of Theorems 1.3 and 1.4. We begin by considering the case of a divisible non-isotropic imaginary root. Recall that a normal variety X with \mathbb{Q} -Cartier canonical divisor K_X is said to have *terminal singularities* if $K_Y = f^*(K_X) + \sum_i a_i E_i$ with $a_i > 0$, where $f : Y \rightarrow X$ is any resolution of singularities, and the sum is over all exceptional divisors of f .

Theorem 6.13. *Let $\alpha \in \Sigma_{\lambda, \theta}$ be an indivisible non-isotropic imaginary root and fix $n \geq 2$ such that $(n, p(\alpha)) \neq (2, 2)$. The symplectic singularity $\mathfrak{M}_\lambda(n\alpha, \theta)$ does not admit a projective symplectic resolution.*

Proof. If $\mathfrak{M}_\lambda(n\alpha, \theta)$ admits a projective symplectic singularity then so too by restriction does the open subset U of Lemma 6.2. Recall from Theorem 1.13 that the singular locus of U is the complement of the open stratum. By Lemma 6.1, this has codimension at least four in U . Therefore, since U has symplectic singularities by Proposition 1.2, [36] says that U has terminal singularities. This implies that if $f : Y \rightarrow U$ is a symplectic resolution then the exceptional locus of f has codimension at least two in Y . On the other hand, we have shown in Corollary 6.8 that U is locally factorial. This implies by van der Waerden purity, see [11, Section 1.40], that the exceptional locus of f is a divisor. This is a contraction. \square

The isomorphism of Theorem 1.3 follows directly from Theorem 3.7. Therefore, it suffices to show that $\mathfrak{M}_\lambda(\alpha, \theta)$ admits a projective symplectic resolution if and only if each $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ admits a projective symplectic resolution. If each $S^{n_i}\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ admits a projective symplectic resolution then clearly $\mathfrak{M}_\lambda(\alpha, \theta)$ also admits a projective symplectic resolution. If $\sigma^{(i)}$ is a real root then $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ is a single point and hence $S^{n_i}\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ trivially admits a symplectic resolution. If $\sigma^{(i)}$ is an isotropic imaginary root then $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ is the partial resolution of a Kleinian singularity. It is well-known that $S^{n_i}\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ admits a projective symplectic resolution in this case. Finally, if $\sigma^{(i)}$ is a non-isotropic imaginary root then $n_i = 1$, since every multiple of a non-isotropic imaginary root $\sigma \in \Sigma_{\lambda, \theta}$ also belongs to $\Sigma_{\lambda, \theta}$. Thus, if each $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ admits a projective symplectic resolution then $\mathfrak{M}_\lambda(\alpha, \theta)$ also admits a projective symplectic resolution.

Finally we must show the converse. That is, we must show that if α contains a non-isotropic imaginary root $\sigma^{(i)}$, with $(\gcd(\sigma^{(i)}), p(\gcd(\sigma^{(i)})^{-1}\sigma^{(i)})) \neq (2, 2)$, in its canonical decomposition, then $\mathfrak{M}_\lambda(\alpha, \theta)$ does not admit a projective symplectic resolution. As observed in the proof of Theorem 6.13, it suffices to show this in a neighborhood of some point. Thus, take $x = (x_1, \dots, x_k) \in \prod_{j=1}^k S^{n_j}\mathfrak{M}_\lambda(\sigma^{(j)}, \theta) = \mathfrak{M}_\lambda(\alpha, \theta)$ such that x_j is in the smooth locus of $S^{n_j}\mathfrak{M}_\lambda(\sigma^{(j)}, \theta)$ for $j \neq i$ and $x_i \in U \subset \mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$. Then, as in the proof of Theorem 6.13, $\mathfrak{M}_\lambda(\alpha, \theta)$ is factorial and terminal at x and hence cannot admit a projective symplectic resolution.

Notice that Theorem 1.4 also follows from the above argument.

6.5. Formal resolutions. Let α be a non-isotropic imaginary root. Though it might not be obvious from Corollary 1.8, the nature of the obstructions to the existence of a projective symplectic resolution of $\mathfrak{M}_\lambda(\alpha, \theta)$ are quite subtle. We have shown that Zariski locally no resolution exists if α is divisible. But then one can ask if a resolution exists étale locally, or in the formal neighborhood of a point? In this section we give a precise answer to this question.

Definition 6.14. The closed point $x \in \mathfrak{M}_\lambda(\alpha, \theta)$ is said to be *formally resolvable* if the formal neighborhood $\widehat{\mathfrak{M}}_\lambda(\alpha, \theta)_x$ of x in $\mathfrak{M}_\lambda(\alpha, \theta)$ admits a projective symplectic resolution.

Lemma 6.15. *If $0 \in \mathfrak{M}_0(\alpha, 0)$ is formally resolvable, then $\mathfrak{M}_0(\alpha, 0)$ also admits a projective symplectic resolution, and conversely.*

Proof. Let \mathbb{C}^\times act on $\text{Rep}(\overline{Q}, \alpha)$ by dilations. Then the moment map μ is homogeneous of degree two and the action of $G(\alpha)$ commutes with the action of \mathbb{C}^\times . This implies that $\mathbb{C}[\mathfrak{M}_\lambda(\alpha, \theta)]$ is an \mathbb{N} -graded, connected algebra. Note also that the Poisson bracket on $\mathbb{C}[\mathfrak{M}_\lambda(\alpha, \theta)]$ has degree -2 . The lemma follows from standard arguments; see [18, Proposition 5.2], [24, Theorem 1.4], and the references therein. The idea is that: 1) The induced action of \mathbb{C}^\times on $\widehat{\mathfrak{M}}_0(\alpha, 0)_0$ lifts to the resolution. 2) The \mathbb{C}^\times -action allows one to globalize the resolution of the formal neighborhood of 0 to a resolution of the whole of $\mathfrak{M}_0(\alpha, 0)$. For the converse statement, we restrict a symplectic resolution of $\mathfrak{M}_0(\alpha, 0)$ to the formal neighborhood of zero. \square

By Corollary 3.3, if one point in a stratum $\mathfrak{M}_\lambda(\alpha, \theta)_\tau \subset \mathfrak{M}_\lambda(\alpha, \theta)$ is formally resolvable, then so too is every other point in the stratum. If $\tau = (e_1, \beta^{(1)}; \dots; e_k, \beta^{(k)})$, then define the greatest common divisor $\gcd(\tau)$ of τ to be the greatest common divisor of the e_i . If the greatest common divisor of τ is k , then each point in $\mathfrak{M}_\lambda(\alpha, \theta)_\tau$ corresponds to a representation of the form $Y^{\oplus k}$ for some polystable representation Y .

Theorem 6.16. *Let $\alpha \in \Sigma_{\lambda, \theta}$ be a non-isotropic root such that $(\gcd(\alpha), p(\gcd(\alpha)^{-1}\sigma)) \neq (2, 2)$. Let $U \subset \mathfrak{M}_\lambda(\alpha, \theta)$ be the union of all strata $\mathfrak{M}_\lambda(\alpha, \theta)_\tau$ such that $\gcd(\tau) = 1$. Then*

- (1) *U is a dense open subset of $\mathfrak{M}_\lambda(\alpha, \theta)$.*
- (2) *The point x is formally resolvable if and only if $x \in U$.*
- (3) *$\mathfrak{M}_\lambda(\alpha, \theta)$ admits a projective symplectic resolution if and only if $U = \mathfrak{M}_\lambda(\alpha, \theta)$.*

Proof. The set U is dense because it contains the open stratum $\mathfrak{M}_\lambda(\alpha, \theta)_{(1, \alpha)}$, consisting of stable representations. We will show that the complement to U is closed in $\mathfrak{M}_\lambda(\alpha, \theta)$. It suffices to show that if the greatest common divisor of ρ is greater than one and $\mathfrak{M}_\lambda(\alpha, \theta)_\tau \subset \overline{\mathfrak{M}_\lambda(\alpha, \theta)_\rho}$ then $\gcd(\tau) > 1$ too. The argument is similar to the proof of Lemma 6.2. Let $x \in \mathfrak{M}_\lambda(\alpha, \theta)_\rho$ and $G_\rho \subset G(\alpha)$ its stabilizer. By Proposition 2.5, there exists $x' \in \mathfrak{M}_\lambda(\alpha, \theta)_\tau$ such that its stabilizer G_τ contains G_ρ . Let $\gcd(\rho) = k$, so that x corresponds to a representation $Y \otimes V$ for some θ -poly-stable representation Y , and k -dimensional vector space V . Notice that $\alpha = k \dim Y$. Then $GL(V)$ is a subgroup of G_ρ , and hence of G_τ too. An elementary argument shows that this implies that x' corresponds to a representation $Y' \otimes V$ for some θ -poly-stable representation Y' . Thus, $\gcd(\tau) > 1$. In fact, we have shown that if $\mathfrak{M}_\lambda(\alpha, \theta)_\tau \subset \overline{\mathfrak{M}_\lambda(\alpha, \theta)_\rho}$, then $\gcd(\rho)$ divides $\gcd(\tau)$. Thus, U is open in $\mathfrak{M}_\lambda(\alpha, \theta)$.

Let $x \in \mathfrak{M}_\lambda(\alpha, \theta)$ have representation type $\tau = (e_1, \beta^{(1)}; \dots; e_k, \beta^{(k)})$, where $k := \gcd(\tau)$. Then, by Corollary 3.3, $\widehat{\mathfrak{M}}_\lambda(\alpha, \theta)_x \simeq \widehat{\mathfrak{M}}_0(\mathbf{e}, 0)_0$. Notice that Lemma 6.15 says that x is formally resolvable if and only if $\mathfrak{M}_0(\mathbf{e}, 0)$ admits a projective symplectic resolution. The greatest common divisor of \mathbf{e} is k . Proposition 4.1 says that \mathbf{e} belongs to $\Sigma_{0,0}$ for the quiver underlying $\mathfrak{M}_0(\mathbf{e}, 0)$. Moreover, by remark 3.4, we have $p(\alpha) = p(\mathbf{e})$ which implies that \mathbf{e} is non-isotropic imaginary. Then part (2) follows from Theorem 6.13 if we can show that

$$(\gcd(\mathbf{e}), p(\gcd(\mathbf{e})^{-1}\mathbf{e})) = (2, 2) \quad \Leftrightarrow \quad (\gcd(\alpha), p(\gcd(\alpha)^{-1}\alpha)) = (2, 2). \quad (8)$$

(Notice that the equality on the right hand side is excluded by the hypothesis of the theorem.) Assume that the left hand side of (8) holds. Then we have $\mathbf{e} = 2\mathbf{e}'$ and $\alpha = 2\alpha'$ for some $\alpha' \in \Sigma_{\lambda, \theta}$. Moreover,

$$4p(\alpha') - 3 = p(\alpha) = p(\mathbf{e}) = 4p(\mathbf{e}') - 3 = 5$$

which implies that $p(\alpha') = 2$. Let $n = \gcd(\alpha')$. Then Lemma 2.2 implies that $\frac{1}{n}\alpha' \in \Sigma_{\lambda, \theta}$ is also non-isotropic imaginary, and hence $p(\frac{1}{n}\alpha') > 1$. But then $p(\alpha) = n^2 p(\alpha') - (n^2 - 1)$ implies that $n = 1$ and hence $(\gcd(\alpha), p(\gcd(\alpha)^{-1}\alpha)) = (2, 2)$.

Finally, for part (3), we note that Theorem 1.4 implies that we just need to show that $U = \mathfrak{M}_\lambda(\alpha, \theta)$ if and only if α is indivisible. But this is obvious. \square

In the case where $\alpha \in \Sigma_{\lambda, \theta}$ is a non-isotropic root such that $(\gcd(\alpha), p(\gcd(\alpha)^{-1}\sigma)) = (2, 2)$, every point in $\mathfrak{M}_\lambda(\alpha, \theta)$ is formally resolvable.

Remark 6.17. If $U \subsetneq \mathfrak{M}_\lambda(\alpha, \theta)$ then Corollary 1.8 implies that any open subset of U not contained in the smooth locus of $\mathfrak{M}_\lambda(\alpha, \theta)$ does not admit a projective symplectic resolution i.e. the singular locus of U consists of points that cannot be resolved Zariski locally, but do admit a resolution in a formal neighborhood (in fact étale locally).

7. NAMIKAWA'S WEYL GROUP

In the paper [37], Namikawa defined a finite group W associated to any conic affine symplectic singularity X , such that the symplectic form on X has weight $\ell > 0$ with respect to the torus action. The group W acts as a reflection group on $H^2(Y, \mathbb{R})$, where $Y \rightarrow X$ is any \mathbb{Q} -factorial terminalization of X , whose existence is guaranteed by the minimal model programme. The group W plays a key role in the birational geometry of X ; see [39] and [2].

One computes W as follows: let \mathcal{L} be a codimension 2 leaf of X and $x \in \mathcal{L}$. Then the formal neighborhood of x in X is isomorphic to the formal neighborhood of 0 in $\mathbb{C}^{2(n-1)} \times \mathbb{C}^2/\Gamma$, where $2n = \dim X$ and $\Gamma \subset SL_2(\mathbb{C})$ is a finite group; see [38, Lemma 1.3]. Associated to Γ , via the McKay correspondence, is a Weyl group $W_{\mathcal{L}}$ of type A, D or E . The fundamental group $\pi_1(\mathcal{L})$ acts on $W_{\mathcal{L}}$ via Dynkin automorphisms. Let $W'_{\mathcal{L}}$ denote the centralizer of $\pi_1(\mathcal{L})$ in $W_{\mathcal{L}}$. Then

$$W := \prod_{\mathcal{L}} W'_{\mathcal{L}}.$$

Thus, in order to compute W , it is essential to classify the codimension 2 leaves of X , and describe $\pi_1(\mathcal{L})$. This is the goal of this section.

7.1. The proof of Theorem 1.17. We assume throughout that $\alpha \in \Sigma_{\lambda, \theta}$, hence it is a root. Therefore the support of α on the quiver is connected. We can assume, up to replacing the quiver by the subquiver whose vertices are the support of α , and whose arrows are the ones with endpoints in the support, that α is sincere. Then, the quiver is connected. We may assume that α is imaginary, otherwise the statement is vacuous.

Our goal is to compute the codimension two leaves of $\mathfrak{M}_\lambda(\alpha, \theta)$, proving Theorem 1.17. Recall from Definition 1.16 that $\alpha = \beta^{(1)} + \dots + \beta^{(s)} + m_1\gamma^{(1)} + \dots + m_t\gamma^{(t)}$ is an isotropic decomposition if

- (1) $\beta^{(i)}, \gamma^{(j)} \in \Sigma_{\lambda, \theta}$.
- (2) The $\beta^{(i)}$ are *pairwise distinct* imaginary roots.
- (3) The $\gamma^{(i)}$ are *pairwise distinct* real roots.
- (4) If \overline{Q}'' is the quiver with $s+t$ vertices without loops and $-(\alpha^{(i)}, \alpha^{(j)})$ arrows between vertices $i \neq j$, where $\alpha^{(i)}, \alpha^{(j)} \in \{\beta^{(1)}, \dots, \beta^{(s)}, \gamma^{(1)}, \dots, \gamma^{(t)}\}$, then Q'' is an affine Dynkin quiver.

- (5) The dimension vector $(1, \dots, 1, m_1, \dots, m_t)$ of Q'' (where there are s one's) equals δ , the minimal imaginary root.

We show that, if we have a stratum of codimension two, we get a isotropic decomposition as above. Consider a stratum of codimension two, of type

$$\tau = \left(n_1, \beta^{(1)}; \dots; n_s, \beta^{(s)}; m_1, \gamma^{(1)}; \dots; m_t, \gamma^{(t)} \right)$$

say. A point of the stratum $\mathfrak{M}_\lambda(\alpha, \theta)_\tau$ corresponds to a θ -polystable representation X ; write it as $\bigoplus_i X_i^{n_i} \oplus \bigoplus_i Y_i^{m_i}$ with the X_i and Y_i distinct θ -stable representations, and $\dim X_i = \beta^{(i)}$ imaginary and $\dim Y_i = \gamma^{(i)}$ real. The $\gamma^{(i)}$ are all distinct since there is at most one simple representation with dimension $\gamma^{(i)}$. The codimension of the stratum is two, therefore

$$1 = p(\alpha) - \sum_i n_i p(\beta^{(i)}) - \sum_i m_i p(\gamma^{(i)}) = p(\alpha) - \sum_i n_i p(\beta^{(i)}), \quad (9)$$

since $\dim \mathfrak{M}_\lambda(\alpha, \theta)_\tau = \sum_i n_i p(\beta^{(i)}) + \sum_i m_i p(\gamma^{(i)})$. Note that the $n_i \beta^{(i)}$ are themselves imaginary roots. Since $\alpha \in \Sigma_{\lambda, \theta}$,

$$p(\alpha) > \sum_i p(n_i \beta^{(i)}) + \sum_i m_i p(\gamma^{(i)}) = \sum_i n_i^2 p(\beta^{(i)}).$$

Therefore, the RHS of (9) is strictly greater than $\sum_i n_i(n_i - 1)p(\beta^{(i)})$. Therefore, if $n_i > 1$ for any i , then the RHS of (9) is strictly greater than one, a contradiction. For the same reason, the $\beta^{(i)}$ are all distinct (otherwise we could group together the $\beta^{(i)}$ that are equal and apply the above combinatorial argument). Thus $n_i = 1$ for all i and the $\beta^{(i)}$ are pairwise distinct.

Let us now set $\alpha^{(i)} := \beta^{(i)}$ for $1 \leq i \leq s$ and $\alpha^{(i)} = \gamma^{(i-s)}$ for $s+1 \leq i \leq s+t$. Similarly let $k_i := 1$ for $1 \leq i \leq s$ and $k_i = m_{i-s}$ for $s+1 \leq i \leq s+t$; let $\mathbf{k} = (k_1, \dots, k_{s+t})$. Note that, for all $i < j$, we must have $(\alpha^{(i)}, \alpha^{(j)}) \leq 0$: this is because $(\alpha^{(i)}, \alpha^{(j)}) = \dim \text{Hom}(Z_i, Z_j) - \dim \text{Ext}(Z_i, Z_j) = -\dim \text{Ext}(Z_i, Z_j)$ where Z_i and Z_j are non-isomorphic θ -stable representations of dimension vectors $\alpha^{(i)}$ and $\alpha^{(j)}$. Therefore we associate two quivers Q' and Q'' to τ . The orientation of the quivers is not uniquely specified (and does not affect what follows). Instead, we describe the double of each quiver. First, let \overline{Q}' be the quiver with $s+t$ vertices, $2p(\alpha^{(i)})$ loops at the i th vertex and $-(\alpha^{(i)}, \alpha^{(j)})$ arrows between i and j . Secondly, as in the definition of isotropic decomposition, Q'' is the quiver obtained from Q' by removing all loops.

Lemma 7.1. *The quiver Q'' is an affine Dynkin quiver and \mathbf{k} is a positive integer multiple of the imaginary root δ .*

Proof. In the proof we use the theory of root systems developed in [23]. Using the fact that $k_i \equiv 1$ when $i \leq s$ and $(\alpha^{(i)}, \alpha^{(i)}) = 2$ for $i > s$, (9) is equivalent to

$$\sum_{i=1}^{s+t} 2k_i^2 + \sum_{i \neq j} k_i k_j (\alpha^{(i)}, \alpha^{(j)}) = 0. \quad (10)$$

Therefore, if $\mathbf{k} = (k_1, \dots, k_{s+t})$ and A is the symmetric matrix with entries $(\alpha^{(i)}, \alpha^{(j)})$ on the (i, j) th off diagonal and all 2s on the diagonal, then $\mathbf{k}^t A \mathbf{k} = 0$ i.e. \mathbf{k} is in the radical of the symmetric form defined by A . Therefore A is a symmetric Cartan matrix in the sense of [23, Section 1.1]. Notice that A is also the matrix that encodes the underlying graph of Q'' i.e. in the notation of *loc. cit.* $S(A)$ is the underlying graph of Q'' .

We claim that Q'' is connected. Assume the contrary. Then there would exist a decomposition $\alpha = \epsilon^{(1)} + \epsilon^{(2)}$ where each of $\epsilon^{(1)}$ and $\epsilon^{(2)}$ is a sum of elements of $\{\beta^{(i)}, m_i \gamma^{(i)}\}$, with each of $\beta^{(i)}$ and $m_i \gamma^{(i)}$ appearing in exactly one of the $\epsilon^{(j)}$, and such that the summands of $\epsilon^{(1)}$ are all orthogonal to the summands of $\epsilon^{(2)}$. Then $p(\epsilon^{(1)}) + p(\epsilon^{(2)}) = p(\alpha) + 1 > p(\alpha)$. We claim that this gives a contradiction to the statement that $\alpha \in \Sigma_{\lambda, \theta}$. If $\epsilon^{(1)}$ and $\epsilon^{(2)}$ were roots, then this would be a contradiction by definition of $\Sigma_{\lambda, \theta}$ since $\epsilon^{(1)}$ and $\epsilon^{(2)}$ both pair to zero with both λ and θ . However, $\epsilon^{(1)}$ and $\epsilon^{(2)}$ are not necessarily roots. Nonetheless, we can take the canonical decompositions $\epsilon^{(c)} = \sum_i \epsilon^{(c, i)}$ with the $\epsilon^{(c, i)} \in \Sigma_{\lambda, \theta}$. Then we will obtain the desired contradiction once we know $p(\epsilon^{(c)}) \leq \sum_i p(\epsilon^{(c, i)})$, which is proved in the following basic lemma (applied to $\alpha = \epsilon^{(c)}$):

Lemma 7.2. *Suppose $\alpha \in \mathbb{N}R_{\lambda, \theta}^+$ has canonical decomposition $\alpha = \sum_i n_i \sigma^{(i)}$ with respect to λ and θ . Then $p(\alpha) \leq \sum_i n_i p(\sigma^{(i)})$.*

Proof. Let λ' be such that $R_{\lambda'}^+ = R_{\lambda, \theta}^+$. As $\alpha \in \mathbb{N}R_{\lambda, \theta}^+ = \mathbb{N}R_{\lambda'}^+$, we know that $\mu_\alpha^{-1}(\lambda')$ is nonempty. The latter is a fiber of a map $\bigoplus_{a \in Q_1} \text{Hom}(\mathbb{C}^{\alpha_{t(a)}}, \mathbb{C}^{\alpha_{h(a)}}) \rightarrow \mathfrak{pg}(\alpha)$, where $\mathfrak{pg}(\alpha)$ is the Lie algebra of $PG(\alpha)$. All of the irreducible components of the latter must have dimension at least $\sum_{a \in \overline{Q_1}} \alpha_{t(a)} \alpha_{h(a)} - \sum_{i \in Q_0} \alpha_i^2 + 1 = \alpha \cdot \alpha - 2\langle \alpha, \alpha \rangle + 1 = \alpha \cdot \alpha + 2p(\alpha) - 1$.

On the other hand, by [6, Theorem 4.4], $\dim \mu_\alpha^{-1}(\lambda') = \alpha \cdot \alpha - \langle \alpha, \alpha \rangle + m = \alpha \cdot \alpha + p(\alpha) + (m - 1)$ where m is the maximum value of $\sum_i p(\alpha^{(i)})$ with $\alpha^{(i)} \in R_{\lambda'}^+$ and $\alpha = \sum \alpha^{(i)}$; as remarked at the top of page 3 in [7], we have $m = \sum_i n_i p(\sigma^{(i)})$ (it is a direct consequence of [7, Theorem 1.1] which we discussed before Theorem 3.7).² We conclude that $\alpha \cdot \alpha + p(\alpha) + (m - 1) \geq \alpha \cdot \alpha + 2p(\alpha) - 1$. Therefore, $m \geq p(\alpha)$, as desired. \square

Remark 7.3. The above implies the following stronger statement: for any decomposition $\alpha = \sum_j \alpha^{(j)}$ with $\alpha^{(j)} \in \mathbb{N}R_{\lambda, \theta}^+$, we have $\sum_j p(\alpha^{(j)}) \leq \sum_i n_i p(\sigma^{(i)})$. This strengthens the statement observed in the proof, from [7, p. 3], in the case where the $\alpha^{(j)}$ are roots. Indeed, for arbitrary $\alpha^{(j)}$, we can apply the lemma to each of the $\alpha^{(j)}$, and then we get that $\sum_j p(\alpha^{(j)}) \leq \sum_j p(\beta^{(j)})$ for some roots $\beta^{(j)} \in R_{\lambda, \theta}^+$ with $\alpha = \sum_j \beta^{(j)}$; then we are back in the case of roots so that $\sum_j n_i p(\sigma^{(i)}) \geq \sum_j p(\beta^{(j)})$.

Remark 7.4. The arguments of [6, 7] can be generalized to the context of the pair (λ, θ) , which as we pointed out in §2.3 would eliminate the need of picking a λ' as in the proof of the lemma above.

²Another interpretation of these facts is that $\mu_\alpha^{-1}(\lambda')$ has some irreducible component of maximum dimension whose generic element is semisimple with the canonical decomposition. The same fact can be deduced for $\mu_\alpha^{-1}(\lambda)^\theta$.

Thus, Q'' is connected. Since \mathbf{k} is in the kernel of A , and \mathbf{k} has strictly positive entries, it follows from [23, Lemma 1.9] that Q'' is a (connected) affine Dynkin quiver and that \mathbf{k} is a multiple of the imaginary root. \square

To conclude one direction of Theorem 1.17, we claim $\mathbf{k} = \delta$. If not, then $s = 0$, since $k_i = 1$ for all $1 \leq i \leq s$. On the other hand, if $s = 0$, then the RHS of (9) is just $p(\alpha)$ itself, so we would conclude that α is isotropic, contradicting our assumptions. Hence, $\mathbf{k} = \delta$.

Conversely, if we have an isotropic decomposition of α , then as remarked before, the stratum $\mathfrak{M}_\lambda(\alpha, \theta)_\tau$, where

$$\tau = (1, \beta^{(1)}; \dots; 1, \beta^{(s)}; m_1, \gamma^{(1)}; \dots; m_t, \gamma^{(t)}),$$

consists of polystable representations of the form $X = \bigoplus_i X_i \oplus \bigoplus_i Y_i^{m_i}$ of $\Pi^\lambda(Q)$ with X_i, Y_j θ -stable and $\dim X_i = \beta^{(i)}$ and $\dim Y_j = \gamma^{(j)}$. We have shown that this stratum has codimension two in $\mathfrak{M}_\lambda(\alpha, \theta)$. The fact that one gets a bijection in this way follows from the fact that each stratum $\mathfrak{M}_\lambda(\alpha, \theta)_\tau$ is connect.

Remark 7.5. Let \mathcal{L} be the leaf labeled by the isotropic decomposition $\alpha = \beta^{(1)} + \dots + \beta^{(s)} + m_1\gamma^{(1)} + \dots + m_t\gamma^{(t)}$ of α . Let B_i denote the fundamental group of the open subset of θ -stable points in $\mathfrak{M}_\lambda(\beta^{(i)}, \theta)$. Then one can check that $\pi_1(\mathcal{L}) = B_1 \times \dots \times B_k$.

8. CHARACTER VARIETIES

Recall from section 1.5 that Σ is a compact Riemannian surface of genus $g > 0$ and π is its fundamental group. We have defined the character varieties

$$\mathcal{Y}(n, g) := \text{Hom}(\pi, \text{SL}(n, \mathbb{C})) // \text{SL}(n, \mathbb{C}), \quad \mathcal{X}(n, g) = \text{Hom}(\pi, \text{GL}(n, \mathbb{C})) // \text{GL}(n, \mathbb{C}).$$

These are affine varieties. Except in the last subsection, we will only consider $\mathcal{X}(n, g)$. Then, in section 8.6 we deduce the corresponding results for $\mathcal{Y}(n, g)$. We begin by recalling the basic properties of the affine varieties $\text{Hom}(\pi, \text{GL})$ and $\mathcal{X}(n, g)$. Based on results of Li [31], as explained in Theorem 2.1 of [15],

Theorem 8.1. (1) Both $\text{Hom}(\pi, \text{GL})$ and $\mathcal{X}(n, g)$ are reduced and irreducible.
(2) $\text{Hom}(\pi, \text{GL})$ is a complete intersection in GL^{2g} .
(3) The generic points of $\text{Hom}(\pi, \text{GL})$ and $\mathcal{X}(n, g)$ correspond to irreducible representations of the fundamental group π .

As shown originally by Goldman [19], the varieties $\mathcal{X}(n, g)$ and $\mathcal{Y}(n, g)$ have a natural Poisson structure. This Poisson structure becomes clear in the realization of these spaces as quasi-Hamiltonian reductions; see [1], where it is shown that the symplectic structure defined by Goldman on the smooth locus of $\mathcal{X}(n, g)$ agrees with the symplectic structure of $\mathcal{X}(n, g)$ as a quasi-Hamiltonian reduction. In particular, if $C_{(1, n)}$ denotes the dense open subset of $\mathcal{X}(n, g)$ parameterizing simple representations of π , then it is shown in [1] that the Poisson structure on $C_{(1, n)}$ is non-degenerate. It

will be useful for us to reinterpret the quasi-Hamiltonian reduction as a moduli space of semi-simple representations of the multiplicative preprojective algebra. Let Q be the quiver with a single vertex and g loops, labeled a_1, \dots, a_g . Let a_i^* denote the loop dual to a_i in the doubled quiver \overline{Q} . Associated to Q is the multiplicative preprojective algebra $\Lambda(Q)$, as defined in [10]. Namely, $\mathbb{C}\overline{Q} \rightarrow \Lambda(Q)$ is the universal homomorphism such that each $1 + a_i a_i^*$ and $1 + a_i^* a_i$ is invertible and

$$\prod_{i=1}^g (1 + a_i a_i^*) (1 + a_i^* a_i)^{-1} = 1.$$

Here the product is ordered. Following [9], let $\Lambda(Q)'$ denote the universal localization of $\Lambda(Q)$, where each a_i is also allowed to be invertible. Let $(T^* \text{Rep}(Q, n))^\circ$ denote the space of all n -dimensional representations (A_i, A_i^*) of $\mathbb{C}\overline{Q}$ such that $1 + A_i A_i^*$, $1 + A_i^* A_i$ and A_i are invertible for all i . It is an open, $\text{GL}(n, \mathbb{C})$ -stable affine subset of $T^* \text{Rep}(Q, n)$. The action of $\text{GL}(n, \mathbb{C})$ on $(T^* \text{Rep}(Q, n))^\circ$ is quasi-Hamiltonian, with multiplicative moment map

$$\Psi : \text{Rep}(\Lambda(Q)', n) \rightarrow \text{GL}, \quad (A_i, A_i^*) \mapsto \prod_{i=1}^g (1 + A_i A_i^*) (1 + A_i^* A_i)^{-1}.$$

As noted in Proposition 2 of [9], the category $\Lambda(Q)'\text{-mod}$ of finite dimensional $\Lambda(Q)'$ -modules is equivalent to $\pi\text{-mod}$, in such a way that we have a GL -equivariant identification

$$\Psi^{-1}(1) \xrightarrow{\sim} \text{Hom}(\pi, \text{GL}), \quad (A_i, A_i^*) \mapsto (A_i, B_i) = (A_i, A_i^{-1} + A_i^*).$$

Hence, we have an identification of Poisson varieties

$$\Psi^{-1}(1) // \text{GL} = \mathcal{X}(n, g).$$

See [42] for further details.

8.1. The space $\mathcal{X}(n, g)$ has a stratification by representation type, which is also the stratification by stabilizer type; see [32, Theorem 5.4]. As in section 6.1, a *weighted partition* ν of n is a sequence $(\ell_1, \nu_1; \dots; \ell_k, \nu_k)$, where $\nu_1 \geq \nu_2 \geq \dots$ and $\sum_{i=1}^k \ell_i \nu_i = n$.

Lemma 8.2. *Assume $n, g > 1$.*

(1) *The strata C_ν of $\mathcal{X}(n, g)$ are labeled by weighted partitions of n such that*

$$\dim C_\nu = 2 \left(k + (g-1) \sum_{i=1}^k \nu_i^2 \right).$$

(2) *If $(n, g) \neq (2, 2)$, then $\dim \mathcal{X}(n, g) - \dim C_\nu \geq 4$ for all $\nu \neq (1, n)$.*

Proof. By Theorem 8.1, the set of points $C_{(1, n)}$ in $\mathcal{X}(n, g)$ parameterizing irreducible representations of π is a dense open subset contained in the smooth locus. Therefore $\dim C_{(1, n)} = 2(1 + n^2(g-1))$. An arbitrary semi-simple representation of π of dimension n has the form $x = x_1^{\oplus \ell_1} \oplus \dots \oplus x_k^{\oplus \ell_k}$, where the x_i are pairwise non-isomorphic irreducible π -modules of dimension ν_i and $n = \sum_{i=1}^k \ell_i \nu_i$. Thus, the representation type strata correspond to weighted partitions of n . Let C_ν denote the

locally closed subvariety of all such representations. If we write the multiset $\{\{\nu_1, \dots, \nu_k\}\}$ as $\{m_1 \cdot \nu_1, \dots, m_r \cdot \nu_r\}$, with $\nu_i \neq \nu_j$, then

$$C_\nu \simeq S^{m_1, \circ} C_{(1, \nu_1)} \times \dots \times S^{m_r, \circ} C_{(1, \nu_r)},$$

where $S^{n, \circ} X$ is the open subset of $S^n X$ consisting of n pairwise distinct points. Thus,

$$\dim C_\nu = \sum_{i=1}^r 2(1 + \nu_i^2(g-1))m_i = 2 \left(k + (g-1) \sum_{i=1}^k \nu_i^2 \right).$$

The second claim is identical to Lemma 6.1 (3). \square

Remark 8.3. Presumably, the stratification of $\mathcal{X}(n, g)$ by representation type is the same as its stratification by symplectic leaves.

Proposition 8.4. *The variety $\mathcal{X}(n, g)$ is normal.*

Proof. The case $g = 1$ follows from Proposition 8.13 below.

We show that the hypotheses of [8, Corollary 7.2] hold in this situation. Notice that since $\text{Hom}(\pi, \text{GL})$ is a complete intersection, it is Cohen-Macaulay. Thus, it satisfies Serre's condition (S_2) . Let $C_{(1, n)} \subset \mathcal{X}(n, g)$ be the open subset of points corresponding to irreducible π -modules. It is contained in the smooth locus of $\mathcal{X}(n, g)$, and hence is normal. Let Z denote its complement. By Lemma 8.2 (2), Z has codimension at least four in $\mathcal{X}(n, g)$ when $(n, g) \neq (2, 2)$. When $(n, g) = (2, 2)$, Z has codimension 2. By [10, Corollary 7.3], if $\zeta : \text{Hom}(\pi, \text{GL}) \rightarrow \mathcal{X}(n, g)$ is the quotient map, then

$$\dim \text{Hom}(\pi, \text{GL}) - \dim \zeta^{-1}(Z) \geq \min_{\nu \neq (1, n)} (\dim \mathcal{X}(n, g) - \dim C_\nu).$$

The right hand side is at least four when $(n, g) \neq (2, 2)$ and is two when $(n, g) = (2, 2)$. Thus, the hypotheses of [8, Corollary 7.2] hold and we conclude that $\mathcal{X}(n, g)$ is normal. \square

Proposition 8.5. *The Poisson variety $\mathcal{X}(n, g)$ is a symplectic singularity.*

Proof. When $g = 1$ the claim follows from Proposition 8.13. The case $(n, g) = (2, 2)$ is dealt with in Corollary 8.12 below.

We assume $g > 1$ and $(n, g) \neq (2, 2)$. We have shown in Proposition 8.4 that the irreducible variety $\mathcal{X}(n, g)$ is normal. By Theorem 8.1, the Poisson structure on the dense open subset $C_{(1, n)}$ of $\mathcal{X}(n, g)$ is non-degenerate. This implies that the Poisson structure on the whole of the smooth locus is non-degenerate since the complement to $C_{(1, n)}$ in $\mathfrak{X}(n, g)$ has codimension at least four. Therefore, since the singular locus of $\mathcal{X}(n, g)$ must also have codimension at least four, it follows from Flenner's Theorem [16] that $\mathcal{X}(n, g)$ has symplectic singularities. \square

8.2. Passage to the tangent cone. In order to study the singularities of $\mathcal{X}(n, g)$, we describe the tangent cone of $\text{Hom}(\pi, \text{GL})$ at a point whose GL -orbit is closed. Let ϕ be such a point and denote by V the corresponding n -dimensional representation of π . Composing ϕ with the adjoint action of GL on $\mathfrak{gl}(V)$, the space $\mathfrak{gl}(V)$ is a π -module. Since Σ is a $K(\pi, 1)$ -space, we have natural identifications

$$\text{Ext}_\pi^i(V, V) = \text{Ext}_\pi^i(\mathbb{C}, \mathfrak{gl}(V)) = H^i(\pi, \mathfrak{gl}(V)) = H^i(\Sigma, \mathcal{V} \otimes \mathcal{V}^\vee),$$

where \mathcal{V} is the local system on Σ corresponding to the π -module V ; see page 59 and Proposition 2.2 of [4]. Cup product in cohomology, followed by the Lie bracket $[-, -] : \mathfrak{gl}(V) \times \mathfrak{gl}(V) \rightarrow \mathfrak{gl}(V)$, defines the *Kuranishi* map

$$\kappa : \text{Ext}_\pi^1(V, V) = \text{Ext}_\pi^1(\mathbb{C}, \mathfrak{gl}(V)) \longrightarrow \text{Ext}_\pi^2(\mathbb{C}, \mathfrak{gl}(V) \otimes \mathfrak{gl}(V)) \longrightarrow \text{Ext}_\pi^2(\mathbb{C}, \mathfrak{gl}(V)) = \text{Ext}_\pi^2(V, V),$$

given by $\varphi \mapsto [\varphi \cup \varphi]$. As shown in [20, Section 4], if $C_V(\text{Hom}(\pi, \text{GL}))$ denotes the tangent cone to V in $\text{Hom}(\pi, \text{GL})$, then there is a $\text{Stab}_{\text{GL}}(V)$ -equivariant isomorphism

$$C_V(\text{Hom}(\pi, \text{GL})) \simeq \kappa^{-1}(0) \subset \text{Ext}_\pi^1(V, V).$$

As explained in [19], the space $\text{Ext}_\pi^1(V, V)$ has a natural symplectic structure, such that the action of $\text{Stab}_{\text{GL}}(V)$ on $\text{Ext}_\pi^1(V, V)$ is Hamiltonian. Decompose the semi-simple representation V as $\bigoplus_{i=1}^k V_i \otimes W_i$, where the V_i are pairwise non-isomorphic simple π -modules. Let \overline{Q} be the quiver with k vertices and $\dim \text{Ext}_\pi^1(V_i, V_j)$ arrows between vertex i and j . Let α be the dimension vector for \overline{Q} given by $\alpha_i = \dim W_i$.

Theorem 8.6. (1) *There is a natural identification $\text{Stab}_{\text{GL}}(V) = G(\alpha)$.*

(2) *The quiver \overline{Q} is the double of some quiver Q .*

(3) *We have a $G(\alpha)$ -equivariant identification $\text{Ext}_\pi^1(V, V) \xrightarrow{\sim} \text{Rep}(\overline{Q}, \alpha)$ of symplectic vector spaces and a $G(\alpha)$ -equivariant identification $\text{Ext}_\pi^2(V, V) \xrightarrow{\sim} \mathfrak{g}(\alpha)$ such that the following diagram is commutative*

$$\begin{array}{ccc} \text{Ext}_\pi^1(V, V) & \xrightarrow{\sim} & \text{Rep}(\overline{Q}, \alpha) \\ \kappa \downarrow & & \downarrow \mu \\ \text{Ext}_\pi^2(V, V) & \xrightarrow{\sim} & \mathfrak{g}(\alpha) \end{array}$$

Proof. The first claim follows directly from the decomposition $\bigoplus_{i=1}^k V_i \otimes W_i$, since $\text{Stab}_{\text{GL}}(V)$ only acts on the W -tensorand.

If \mathcal{V}_i denotes the irreducible local system on Σ corresponding to V_i , then we have natural identifications $\text{Ext}_\pi^1(V_i, V_j) = H^1(\Sigma, \mathcal{V}_i \otimes \mathcal{V}_j^\vee)$ and $\text{Ext}_\pi^2(V_i, V_j) = H^2(\Sigma, \mathcal{V}_i \otimes \mathcal{V}_j^\vee)$ imply by Poincaré-Verdier duality (see [12, Corollary 3.3.12]) that

- $\text{Ext}_\pi^2(V_i, V_i) \simeq \mathbb{C}$ and $\text{Ext}_\pi^2(V_i, V_j) = 0$ for $i \neq j$.

- The cup product defines a non-degenerate pairing

$$\langle -, - \rangle : \text{Ext}_\pi^1(V_i, V_j) \times \text{Ext}_\pi^1(V_j, V_i) \rightarrow \mathbb{C}.$$

The existence of the non-degenerate pairing implies that $\dim \text{Ext}_\pi^1(V_i, V_j) = \dim \text{Ext}_\pi^1(V_j, V_i)$ when $i \neq j$. Moreover, each $\text{Ext}_\pi^1(V_i, V_i)$ is a symplectic vector space [19], and hence $\dim \text{Ext}_\pi^1(V_i, V_i)$ is even. Thus, \overline{Q} is the double of some quiver Q , confirming (2).

Finally, we have $G(\alpha)$ -equivariant identifications

$$\text{Ext}_\pi^1(V, V) = \bigoplus_{i,j} \text{Ext}_\pi^1(V_i, V_j) \otimes \text{Hom}(W_i, W_j) = \text{Rep}(\overline{Q}, \alpha)$$

and

$$\text{Ext}_\pi^2(V, V) = \bigoplus_{i,j} \text{Ext}_\pi^2(V_i, V_j) \otimes \text{Hom}(W_i, W_j) = \bigoplus_{i=1}^k \text{End}(W_i),$$

since $\text{Ext}_\pi^2(V_i, V_j) = 0$ for $i \neq j$. Now view the quadratic map $\kappa : \text{Ext}_\pi^1(V, V) \rightarrow \text{Ext}_\pi^2(V, V)$ as a linear one, $\text{Ext}_\pi^1(V, V) \otimes \text{Ext}_\pi^1(V, V) \rightarrow \text{Ext}_\pi^2(V, V)$. This map can be written as $a \otimes b \mapsto a \circ b - b \circ a$, where the map \circ is the usual composition,

$$\circ : \text{Ext}_\pi^1(V_i \otimes W_i, V_j \otimes W_j) \times \text{Ext}_\pi^1(V_j \otimes W_j, V_k \otimes W_k) \rightarrow \text{Ext}_\pi^2(V_i, V_k) \otimes \text{Hom}(W_i, W_k),$$

which is only nonzero when $i = k$. For $i = k$, if we write $\text{Ext}_\pi^1(V_i \otimes W_i, V_j \otimes W_j) = \text{Ext}_\pi^1(V_i, V_j) \otimes \text{Hom}(W_i, W_j)$, the above map becomes the tensor product of the symplectic pairing between $\text{Ext}_\pi^1(V_i, V_j)$ and $\text{Ext}_\pi^1(V_j, V_i)$ and the composition on Hom spaces. Thus, linearly extending to V , we obtain the usual moment map μ , as required. \square

Luna's slice theorem implies that:

Corollary 8.7. *The tangent cone to $[V] \in \mathcal{X}(n, g)$ is isomorphic to $\mathfrak{M}_0(\alpha, 0)$ for the quiver Q and dimension vector α described above.*

Question 8.8. When is the formal neighborhood of $[V]$ in $\mathcal{X}(n, g)$ isomorphic to the formal neighborhood of 0 in $\mathfrak{M}_0(\alpha, 0)$?

Since the latter is a cone, the question is equivalent to asking when the formal neighborhood of $[V]$ in $\mathcal{X}(n, g)$ is conical. This is true in the case V is the trivial n -dimensional representation of π . Note that, equivalently to Question 8.8, we can ask the Koszul dual question: Is $\text{RHom}_\pi(V, V)$, together with the Kuranishi bracket (i.e., the commutator under cup product), a formal dglg? This would be implied if $\text{RHom}_\pi(V, V)$ were a formal dga.

Lemma 8.9. *The singular locus of $\mathcal{X}(n, g)$ is the closed subset consisting of non-simple representations. Its irreducible components are labeled by integers $1 \leq n' \leq n/2$.*

Proof. The proof is identical to the proof of [26, Proposition 6.1]. Theorem 8.6 implies that if the point $x \in \mathcal{X}(n, g)$ corresponds to a simple representation V , then x is smooth. For each $1 \leq n' \leq n/2$, let $\varphi(n') : \mathcal{X}(n', g) \times \mathcal{X}(n - n', g) \rightarrow \mathcal{X}(n, g)$ denote the map $([V_1], [V_2]) \mapsto [V_1 \oplus V_2]$. It is a finite morphism. Clearly, every semi-simple, but not simple, π -module of dimension n lies in the image of some $\varphi(n')$. Also, $\text{Im } \varphi(n') \cap \text{Im } \varphi(n'')$ is a proper subset of $\text{Im } \varphi(n')$ for all $n' \neq n''$ since a generic point of $\text{Im } \varphi(n')$ is the direct sum of exactly two simple modules. Therefore the $\text{Im } \varphi(n')$ are precisely the irreducible components of the complement to the open subset of simple representations. Thus, it suffices to show that the generic point of $\text{Im } \varphi(n')$ is singular in $\mathcal{X}(n, g)$. Such a generic point is $[V_1 \oplus V_2]$, where V_1 and V_2 are simple π -modules of dimension n' and $n - n'$ respectively.

It suffices to show that the tangent cone at this point is singular. By Corollary 8.7, the tangent cone is isomorphic to $0 \in \mathfrak{M}_0(\alpha, 0)$ for some quiver Q and dimension vector α . In this case, we get the quiver Q with $\frac{1}{2} \dim \text{Ext}_\pi^1(V_1, V_1)$ loops at vertex 1, $\frac{1}{2} \dim \text{Ext}_\pi^1(V_2, V_2)$ loops at vertex 2 and $\dim \text{Ext}_\pi^1(V_1, V_2)$ arrows from vertex 1 to vertex 2. The dimension vector is $\alpha = (1, 1)$. The space $\mathfrak{M}_0(\alpha, 0)$ is singular if and only if $\dim \text{Ext}_\pi^1(V_1, V_2) > 1$ (removing the loops, which do not contribute to the singularities, $\mathfrak{M}_0(\alpha, 0)$ is isomorphic to the closure of the minimal nilpotent orbit in \mathfrak{gl}_n , where $n = \dim \text{Ext}_\pi^1(V_1, V_2)$). Since V_1 and V_2 are not isomorphic, [10, Theorem 1.6] implies that

$$\dim \text{Ext}_\pi^1(V_1, V_2) = (2g - 2)n'(n - n') > 1$$

as required. \square

Remark 8.10. We note that [10, Theorem 1.6] allows one to easily compute the Euler characteristic of local systems on compact Riemann surfaces. For instance, it implies that if \mathcal{L} is an irreducible local system on Σ then

$$\chi(\mathcal{L}) = (2g - 2) \text{rk}(\mathcal{L}).$$

Presumably, this is well known to experts.

8.3. The case $(n, g) = (2, 2)$. The case $(n, g) = (2, 2)$ can be thought of as a “local model” for the moduli space M_{2v} of semi-stable shaves with Mukai vector $2v$ on an abelian or $K3$ surface, where v is primitive, such that $\langle v, v \rangle = 2$. Therefore we are able to apply directly the results of Lehn and Sorger [30] in this case. Lemma 8.2 (1) says that $\mathcal{X}(2, 2)$ has three strata, $C_{(1,2)}$ consisting of simple representations E , $C_{(1,1;1,1)}$ consisting of semi-simple representations $E = F_1 \oplus F_2$, where F_1 and F_2 are a pair of non-isomorphic one-dimensional representations of π , and $C_{(2,1)}$ the stratum of semi-simple representations $E = F^{\oplus 2}$, where F is a one-dimensional representation. By Corollary 8.9, the singular locus of $\mathcal{X}(2, 2)$ equals $\overline{C}_{(1,1;1,1)} = C_{(1,1;1,1)} \sqcup C_{(2,1)}$.

Theorem 8.11 (Lehn-Sorger, [30]). *The blowup $\sigma : \tilde{\mathcal{X}}(2, 2) \rightarrow \mathcal{X}(2, 2)$ along the reduced ideal defining the singular locus of $\mathcal{X}(2, 2)$ defines a semi-small resolution of singularities.*

Proof. We sketch the proof, based on the results in [30]. Fix a point $E \in C_{(1,1;1,1)}$ and $E' \in C_{(2,1)}$. Theorem 8.6 says that the tangent cone $C_E(\mathcal{X}(2,2))$ is isomorphic to $\mathbb{C}^8 \times (\mathbb{C}^2/\mathbb{Z}_2)$ and the tangent cone $C_{E'}(\mathcal{X}(2,2))$ is isomorphic to $\mathbb{C}^4 \times \mathcal{N}$, where \mathcal{N} is the orbit closure in $\mathfrak{sp}(4)$ defined in section 5.1. The proof of [30, Théorème 4.5] goes through word for word in this situation (one has to check that Propositions A.1 and A.2 of the appendix to *loc. cit.* hold in this setting), and we deduce that there are isomorphisms of analytic germs

$$(\mathcal{X}(2,2), E) \simeq (\mathbb{C}^8 \times (\mathbb{C}^2/\mathbb{Z}_2), 0), \quad (\mathcal{X}(2,2), E') \simeq (\mathbb{C}^4 \times \mathcal{N}, 0).$$

(The first isomorphism follows from [38, Lemma 1.3]). Clearly, blowing up $\mathbb{C}^8 \times (\mathbb{C}^2/\mathbb{Z}_2)$ along the singular locus gives a semi-small resolution of singularities. The key result [30, Théorème 2.1] says that blowing up along the reduced ideal defining the singular locus in $\mathbb{C}^4 \times \mathcal{N}$ also produces a semi-small resolution of singularities. \square

Corollary 8.12. *The blowup $\tilde{\mathcal{X}}(2,2)$ of $\mathcal{X}(2,2)$ along the reduced ideal defining the singular locus of $\mathcal{X}(2,2)$ is a symplectic manifold and $\tilde{\mathcal{X}}(2,2)$ has symplectic singularities.*

Proof. Let $\sigma : \tilde{\mathcal{X}}(2,2) \rightarrow \mathcal{X}(2,2)$ denote the blowup map. The singularities of $\mathcal{X}(2,2)$ in an analytic neighborhood of a point in $C_{(1,1;1,1)}$ are equivalent to an A_1 singularity. Therefore the pullback $\sigma^*\omega$ of the symplectic 2-form ω on the smooth locus of $\mathcal{X}(2,2)$ extends to a symplectic 2-form on $\sigma^{-1}(U)$, where U is the open set $C_{(1,2)} \cup C_{(1,1;1,1)}$. Since σ is semi-small, $\sigma^{-1}(C_{(2,1)})$ has codimension at least 3 in $\tilde{\mathcal{X}}(2,2)$. Therefore, $\sigma^*\omega$ extends to a symplectic 2-form on the whole of $\tilde{\mathcal{X}}(2,2)$. Since we have shown in Proposition 8.4 that $\mathcal{X}(2,2)$ is normal, Lemma 6.11 implies that $\mathcal{X}(2,2)$ has symplectic singularities. \square

8.4. The genus one case. Let G be either GL or SL and \mathbb{T} a maximal torus in G . The following is well-known. It can be deduced from the corresponding statement for the commuting variety in $\mathfrak{g} \times \mathfrak{g}$; see [17, Sections 2.7 and 2.8]

Proposition 8.13. *Fix $g = 1$. As symplectic singularities, the G -character variety of Σ is isomorphic to $(\mathbb{T} \times \mathbb{T})/\mathfrak{S}_n$.*

Unlike the case $g > 1$, it is not clear whether $\text{Hom}(\pi, G)$ is reduced, but it is shown in [17] that the corresponding G -character variety is reduced. In the case $G = GL$, the Hilbert-Chow morphism defines a symplectic resolution $\pi : \text{Hilb}^n(\mathbb{C}^\times \times \mathbb{C}^\times) \rightarrow (\mathbb{T} \times \mathbb{T})/\mathfrak{S}_n$. Similarly, the preimage $\text{Hilb}_0^n(\mathbb{C}^\times \times \mathbb{C}^\times) \subset \text{Hilb}^n(\mathbb{C}^\times \times \mathbb{C}^\times)$ of $\mathcal{Y}(n,1) \subset \mathcal{X}(n,1)$ under π defines a symplectic resolution of $\mathcal{Y}(n,1)$; for want of a better name, we call $\text{Hilb}_0^n(\mathbb{C}^\times \times \mathbb{C}^\times)$ the *barycentric Hilbert scheme*. Notice that the case $n = 1$ is trivial since $\mathcal{X}(1,1) = \mathbb{C}^\times \times \mathbb{C}^\times$ with its standard symplectic structure.

8.5. Local factorality. Recall that $\zeta : \text{Hom}(\pi, GL) \rightarrow \mathcal{X}(n,g)$ is the quotient map. The action of GL on $\text{Hom}(\pi, GL)$ factors through PGL . The open subset of $\text{Hom}(\pi, GL)$ where PGL acts freely is denoted $\text{Hom}(\pi, GL)_{\text{free}}$.

Lemma 8.14. *Assume that $g > 1$ and $(n, g) \neq (2, 2)$. The variety $\text{Hom}(\pi, \text{GL})$ is normal and locally factorial. Moreover, the complement to $\text{Hom}(\pi, \text{GL})_{\text{free}}$ in $\text{Hom}(\pi, \text{GL})$ has codimension at least four.*

Proof. As noted previously, $\text{Hom}(\pi, \text{GL})$ is a complete intersection and hence Cohen-Macaulay. Thus, it satisfies (S_2) . By a theorem of Grothendieck, [26, Theorem 3.12], in order to show that $\text{Hom}(\pi, \text{GL})$ is locally factorial, it suffice to check that it satisfies (R_3) too. But this follows from the proof of Proposition 8.4, where it was shown that $\dim \text{Hom}(\pi, \text{GL}) - \dim \zeta^{-1}(Z) \geq 4$. Since $\text{Hom}(\pi, \text{GL}) \setminus \zeta^{-1}(Z) \subset \text{Hom}(\pi, \text{GL})_{\text{free}}$, this also implies that the complement to $\text{Hom}(\pi, \text{GL})_{\text{free}}$ in $\text{Hom}(\pi, \text{GL})$ has codimension at least four. \square

In this context, Lemma 6.5 says that every PGL -equivariant line bundle on $\text{Hom}(\pi, \text{GL})_{\text{free}}$ extends to a PGL -equivariant line bundle on $\text{Hom}(\pi, \text{GL})$. Let $\mathcal{X}(n, g)_{\text{free}}$ denote the image of $\text{Hom}(\pi, \text{GL})_{\text{free}}$ in $\mathcal{X}(n, g)$. We can repeat Drezet's arguments, as written in Theorem 6.6, word for word for $\mathcal{X}(n, g)$. His theorem says: Let $x \in \mathcal{X}(n, g)$ and y a lift in $\text{Hom}(\pi, \text{GL})$. The following are equivalent:

- (i) The local ring $\mathcal{O}_{\mathcal{X}(n, g), x}$ is a unique factorization domain.
- (ii) For every line bundle M_0 on $\mathcal{X}(n, g)_{\text{free}}$, there exists an open subset $U \subset \mathcal{X}(n, g)$ containing both x and $\mathcal{X}(n, g)_{\text{free}}$ such that M_0 extends to a line bundle M on U .
- (iii) For every PGL -equivariant line bundle L on $\text{Hom}(\pi, \text{GL})$, the stabilizer of y acts trivially on the fiber L_y .

Theorem 8.15. *Assume that $g > 1$ and $(n, g) \neq (2, 2)$. Then, the variety $\mathcal{X}(n, g)$ has locally factorial, terminal singularities.*

Proof. The stratum C_ρ of type $\rho = (n, 1)$ is contained in the closure of all other strata in $\mathcal{X}(n, g)$ (this can be proven by induction on n using the morphisms $\varphi(n')$ defined in the proof of Lemma 8.9). If y is a lift in $\text{Hom}(\pi, \text{GL})$ of a point of C_ρ then y corresponds to the representation $\mathbb{C}^{\oplus n}$, where \mathbb{C} denotes here the trivial π -module. Therefore $\text{PGL}_y = \text{PGL}$ has no non-trivial characters. In particular, PGL_y will act trivially on L_y for any PGL -equivariant line bundle on $\text{Hom}(\pi, \text{GL})$. Hence, we deduce from Theorem 6.6 that $\mathcal{X}(n, g)$ is factorial at every point of C_ρ .

Now consider an arbitrary stratum C_τ in $\mathcal{X}(n, g)$. If $\mathcal{X}(n, g)$ is factorial at one point of the stratum then it will be factorial at every point in the stratum. On the other hand, the main result of [3] says that the subset of factorial points of $\mathcal{X}(n, g)$ is an open subset. Since this open subset is a union of strata and contains the unique closed stratum, it must be the whole of $\mathcal{X}(n, g)$. \square

Arguing as in the proof of Theorem 6.13, Theorem 8.15 implies:

Corollary 8.16. *Assume that $g > 1$ and $(n, g) \neq (2, 2)$. Then $\mathcal{X}(n, g)$ does not admit a symplectic resolution.*

Remark 8.17. Presumably, a similar analysis can be done in order to classify which moduli spaces of semi-simple representations of an arbitrary multiplicative deformed preprojective algebra admit symplectic resolutions. We leave this to the interested reader.

8.6. The SL-character variety. Recall that $\mathcal{Y}(n, g)$ is the character variety associated to the compact Riemann surface Σ , of genus g , with values in $\mathrm{SL}(n, \mathbb{C})$. Let $T \simeq (\mathbb{C}^\times)^{2g}$ denote the $2g$ -torus.

Lemma 8.18. *The character variety $\mathcal{X}(n, g)$ is an étale locally trivial fiber bundle over T with fiber $\mathcal{Y}(n, g)$.*

Proof. Let $\varrho : \mathrm{Hom}(\pi, \mathrm{GL}) \rightarrow T$ be the map sending (A_i, B_i) to $(\det(A_i), \det(B_i))$. This map is GL -equivariant, where the action on T is trivial. Moreover, it fits into a commutative diagram of GL -varieties

$$\begin{array}{ccc} \mathrm{Hom}(\pi, \mathrm{SL}) \times_{\mathbb{Z}_n^{2g}} T & \xrightarrow{\sim} & \mathrm{Hom}(\pi, \mathrm{GL}) \\ \mathrm{pr} \searrow & & \swarrow \varrho \\ & T & \end{array} \quad (11)$$

where \mathbb{Z}_n^{2g} acts freely on T , and the map $\mathrm{Hom}(\pi, \mathrm{SL}) \times T \rightarrow \mathrm{Hom}(\pi, \mathrm{GL})$ sends $((A_i, B_i), (t_i, s_i))$ to $(t_i A_i, s_i B_i)$. Therefore it descends to a commutative diagram

$$\begin{array}{ccc} \mathcal{Y}(n, g) \times_{\mathbb{Z}_n^{2g}} T & \xrightarrow{\sim} & \mathcal{X}(n, g) \\ \mathrm{pr} \searrow & & \swarrow \varrho \\ & T & \end{array} \quad (12)$$

where \mathbb{Z}_n^{2g} acts freely on $\mathcal{Y}(n, g) \times T$. □

Proof of Theorem 1.23. Let I denote the reduced ideal in $\mathbb{C}[\mathcal{Y}(2, 2)]$ defining the singular locus. Since the singular locus is stable under the action of \mathbb{Z}_n^{2g} so too is I . Therefore, the action of \mathbb{Z}_n^{2g} lifts to the blowup $\tilde{\mathcal{Y}}(2, 2)$ making $\sigma : \tilde{\mathcal{Y}}(2, 2) \rightarrow \mathcal{Y}(2, 2)$ equivariant. Theorem 8.11, together with the fact that

$$\tilde{\mathcal{X}}(2, 2) \simeq \tilde{\mathcal{Y}}(2, 2) \times_{\mathbb{Z}_n^{2g}} T,$$

implies that $\tilde{\mathcal{Y}}(2, 2)$ is smooth. Moreover, the fact that $\tilde{\mathcal{X}}(2, 2) \rightarrow \mathcal{X}(2, 2)$ is semi-small implies that $\sigma : \tilde{\mathcal{Y}}(2, 2) \rightarrow \mathcal{Y}(2, 2)$ is semi-small. The argument that this implies that σ is a symplectic resolution is identical to the first part of the proof of Corollary 8.12. □

Proof of Theorem 1.18. Proposition 8.13 implies that Theorem 1.18 holds when $g = 1$. When $(n, g) = (2, 2)$, Theorem 1.23 and Proposition 6.11 imply that $\mathcal{Y}(2, 2)$ has symplectic singularities. Therefore we assume that $g > 1$ and $(g, n) \neq (2, 2)$.

We begin by showing that $\mathcal{Y}(n, g)$ is normal. Lemma 8.18 implies that $\mathcal{Y}(n, g)$ is an irreducible variety of dimension $2(g-1)(n^2-1)$ since $\dim \mathcal{X}(n, g) = 2n^2(g-1) + 2$. If $\mathcal{Y}(n, g)$ were not normal,

then $\mathcal{Y}(g, n) \times T$ would also not be normal. But the fact that $\mathcal{Y}(n, g) \times_{\mathbb{Z}_n^{2g}} T \simeq \mathcal{X}(n, g)$ is normal, and the map $\mathcal{Y}(n, g) \times T \rightarrow \mathcal{X}(n, g)$ is étale, implies by [33, Proposition 3.17] that $\mathcal{Y}(n, g) \times T$ is normal. Thus, $\mathcal{Y}(n, g)$ is normal. The identification $\mathcal{Y}(n, g) \times_{\mathbb{Z}_n^{2g}} T \simeq \mathcal{X}(n, g)$ of Lemma 8.18 is Poisson, where we equip $\mathcal{Y}(n, g) \times T$ with the product Poisson structure. We deduce that the Poisson structure on the smooth locus of $\mathcal{Y}(n, g)$ is non-degenerate, and the singular locus of $\mathcal{Y}(n, g)$ has codimension at least 4 when $(n, g) \neq (2, 2)$. Therefore, repeating the proof of Proposition 8.5, we deduce that $\mathcal{Y}(n, g)$ has symplectic singularities. \square

Proof of Theorem 1.19. Recall that we have assumed that $g > 1$ and $(n, g) \neq (2, 2)$.

As noted in the proof of Theorem 1.18, $\mathcal{Y}(n, g)$ is a symplectic singularity whose singular locus has codimension at least 4. Therefore $\mathcal{Y}(n, g)$ has terminal singularities. To show that $\mathcal{Y}(n, g)$ is locally factorial, one simply repeats word for word the arguments of section 8.5, but with GL replaced by SL throughout, and using diagrams (11) and (12) to deduce the required dimension inequalities. \square

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